THE POINTS ON A SHIMURA VARIETY MODULO A PRIME OF GOOD REDUCTION

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ABSTRACT. We explain, in the case of good reduction, the conjecture of Langlands and Rapoport describing the structure of the points on the reduction of a Shimura variety (Langlands and Rapoport 1987, 5.e, p169), and we derive from it the formula conjectured in (Kottwitz 1990, 3.1), which expresses a certain trace as a sum of products of (twisted) orbital integrals. Also we introduce the notion of an *integral* canonical model for a Shimura variety, and we extend the conjecture of Langlands and Rapoport to Shimura varieties defined by groups whose derived group is not simply connected. Finally, we briefly review Kottwitz's stabilization of his formula.

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0. INTRODUCTION

After giving an outline of the article, we discuss some of the history of the problems described here. Then we make some general comments, and list some of our notations.

Outline. In §1 we review the basic theory of Shimura varieties, and in particular, the notion of the canonical model $\mathrm{Sh}(G,X)$ of a Shimura variety over its reflex field $E = E(G,X)$. Let K_p be a compact open subgroup of $G(\mathbb{Q}_p)$, and let

$$\mathrm{Sh}_p(G,X) = \mathrm{Sh}(G,X)/K_p.$$

If v is a prime of E lying over p, then $\mathrm{Sh}_p(G,X)$ may fail to have good reduction at v for essentially two different reasons: the group G may be ramified at p, or K_p may not be maximal. The assumption that K_p is hyperspecial obviates both problems, and Langlands [Langlands 1976, p411] suggests that $\mathrm{Sh}_p(G,X)$ will then have a smooth model over the ring of integers $\mathcal{O}_v$ in E_v. But if the Shimura variety has one smooth model, it will have many, and their points with coordinates in $\mathbb{F}$ may differ. In §2 we introduce the notion of an ***integral*** canonical model of a Shimura variety. This is a smooth model of $\mathrm{Sh}_p(G,X)$ over $\mathcal{O}_v$ satisfying certain conditions sufficient to determine it uniquely. To check that the definition is reasonable, we verify that the moduli schemes constructed in [Mumford 1965] form an integral canonical model for the Siegel modular variety.

Henceforth, we assume that our Shimura variety has a canonical integral model $\mathrm{Sh}_p(G,X)_v$, and we write $\mathrm{Sh}_p(\mathbb{F})$ for the set of its points with coordinates in $\mathbb{F}$. There are commuting actions of the geometric Frobenius element Φ and of $G(\mathbb{A}_f^p)$ on $\mathrm{Sh}_p(\mathbb{F})$, and the purpose of the conjecture of Langlands and Rapoport is to describe the isomorphism class of the triple $(\mathrm{Sh}_p(\mathbb{F}), \Phi, \times)$, that is, of the set with the two commuting actions.

In §3 we define a groupoid $\mathfrak{P}$ with additional structure, the ***pseudomotivic groupoid,*** that (conjecturally) is the groupoid associated with the Tannakian category of motives over the algebraic closure $\mathbb{F}$ of a finite field.

Let $\mathfrak{G}_G$ be the neutral groupoid defined by G, and consider a homomorphism $\varphi\colon \mathfrak{G}_G \to \mathfrak{P}$. In §4 we explain how to attach to φ a triple $(S(\varphi), \Phi(\varphi), \times(\varphi))$, where $S(\varphi)$ is a set of the form

$$S(\varphi) = I_\varphi(\mathbb{Q})^- \backslash X^p(\varphi) \times X_p(\varphi),$$

$\Phi(\varphi)$ is a "Frobenius" operator, and $\times(\varphi)$ is an action of $G(\mathbb{A}_f^p)$ on $S(\varphi)$ commuting with the action of $\Phi(\varphi)$. The main conjecture (see 4.4) then states that

$$(\mathrm{Sh}_p(\mathbb{F}), \Phi, \times) \approx \textstyle\coprod_\varphi (S(\varphi), \Phi(\varphi), \times(\varphi)) \tag{0.1}$$

where the disjoint union is over a certain set of isomorphism classes of homomorphisms $\varphi\colon \mathfrak{P} \to \mathfrak{G}_G$.

In our statement of this conjecture, we have changed the indexing set for the φ's from that of Langlands and Rapoport so that the conjecture now applies also to Shimura varieties $\mathrm{Sh}(G,X)$ for which G^{der} is not simply connected. An example [Langlands and Rapoport 1987, §7] shows that the original conjecture fails in general when G^{der} is not simply connected. Here we turn their arguments around to show that if the (modified) conjecture is true for the Shimura varieties for which G^{der} is simply connected, then it is true for all Shimura varieties.

The philosophy underlying Deligne's axioms for a Shimura variety is that a Shimura variety with rational weight should be a moduli varieties for motives, and it was this that suggested the general shape of (0.1). The realization of $\mathrm{Sh}(G,X)$ as a moduli variety for motives depends on the choice of a faithful representation $\xi\colon G \hookrightarrow \mathrm{GL}(V)$ of G. Choose such a ξ and let $\mathfrak{t}$ be a family of tensors for V such that $\xi(G)$ is the subgroup of $\mathrm{GL}(V)$ fixing the tensors (up to a constant). There should be a bijection between $\mathrm{Sh}_p(\mathbb{F})$ and a certain set of isomorphism classes of quadruples $(M,\mathfrak{s},\eta^p,\Lambda_p)$ with M a motive over $\mathbb{F}$, $\mathfrak{s}$ a set of tensors for M, η^p a prime-to-p level structure on M (an isomorphism $V(\mathbb{A}_f^p) \to M_f^p$), and Λ_p a p-integral structure on M. The isomorphism classes of pairs $(M,\mathfrak{s})$ should be in one-to-one correspondence with the isomorphism classes of "admissible" φ's, and if φ corresponds to $(M,\mathfrak{s})$, then $X^p(\varphi)$ should be the set of "admissible" prime-to-p level structures on M, and $X_p(\varphi)$ should be the set of "admissible" p-integral structures on M. Since $I_\varphi(\mathbb{Q})$ is the group of automorphisms of $(M,\mathfrak{s})$, when we consider the quadruples up to isomorphism, we find that

$$S(\varphi) = I_\varphi(\mathbb{Q})\backslash(X^p(\varphi)/Z') \times X_p(\varphi),$$

where Z' is the closure of $Z(\mathbb{Q})$ in $Z(\mathbb{A}_f^p)$.

Let $K = K^p \cdot K_p$ be a compact open subgroup of $G(\mathbb{A}_f)$, and let $\mathrm{Sh}_K(G,X) = \mathrm{Sh}(G,X)/K$. In §5 we derive from (0.1) a description of the set $\mathrm{Sh}_K(\mathbb{F}_q)$ of points on $\mathrm{Sh}_K(G,X)_v$ with coordinates in a finite field $\mathbb{F}_q$ containing the residue field $\kappa(v)$ at v. We leave it as an easy exercise in combinatorics for the reader to show that the knowledge of $\mathrm{Card}(\mathrm{Sh}_K(\mathbb{F}_q))$ for all $\mathbb{F}_q \supset \kappa(v)$ is equivalent to the knowledge of the pair $(\mathrm{Sh}_K(\mathbb{F}),\Phi)$ (up to isomorphism), that is, without the action[1] of $G(\mathbb{A}_f^p)$.

Knowing the cardinality of $\mathrm{Sh}_K(\mathbb{F}_q)$ for all $\mathbb{F}_q \supset \kappa(v)$ is equivalent to knowing that part of the local zeta function of $\mathrm{Sh}_K(G,X)$ at v coming from the cohomology

[1] Ironically, because it is the main point of their paper, Langlands and Rapoport misstate their conjecture by not requiring that the bijection (0.1) be $G(\mathbb{A}_f^p)$-equivariant—Langlands assures me that this should be considered part of the conjecture. It is essential for the applications.

of $\mathrm{Sh}_K(G,X)$ with compact support (hence the whole of the zeta function when $\mathrm{Sh}(G,X)$ is complete). More generally, when we want to study the zeta function of the sheaf $\mathcal{V}(\xi)$ on $\mathrm{Sh}(G,X)$ defined by a representation ξ of G, we need to consider a sum

$$\textstyle\sum_{t'} \mathrm{Tr}(\mathcal{T}(g)^{(r)}|\mathcal{V}_t(\xi)) \tag{0.2}$$

where $\mathcal{T}(g)$ is the Hecke operator defined by $g \in G(\mathbb{A}_f^p)$, $\mathcal{T}(g)^{(r)}$ denotes the composite of the Hecke correspondence defined by g with the rth power of the Frobenius correspondence, and the sum runs over the points of t' of $\mathrm{Sh}_{K\cap gKg^{-1}}(G,X)(\mathbb{F})$ such that $\mathcal{T}(g)(t') = \Phi^r(t') = t$ (see C.6). In §6 we derive from the main conjecture a formula

$$\textstyle\sum_{t'} \mathrm{Tr}(\mathcal{T}(g)^{(r)}|\mathcal{V}_t(\xi)) = \sum_{(\varphi,\varepsilon)} \mathrm{vol}(I_\varphi(\mathbb{Q})\backslash I_\varphi(\mathbb{A}_f)) \cdot O_\gamma(f^p) \cdot TO_\delta(\phi_r) \cdot \mathrm{Tr}(\gamma_0) \tag{0.3}$$

where $O_\gamma(f^p)$ is a certain orbital integral, $TO_\delta(\phi_r)$ is a certain twisted orbital integral, and $(\gamma_0;\gamma,\delta)$ is a certain triple attached to a pair (φ,ε) in the indexing set. Call a triple $(\gamma_0;\gamma,\delta)$ ***effective*** if it arises from a pair (φ,ε). Then each effective triple occurs only finitely number many times, and the term corresponding to (φ,ε) depends only on the triple attached to it. Thus we can rewrite (0.3) as

$$\textstyle\sum_{t'} \mathrm{Tr}(\mathcal{T}(g)^{(r)}|\mathcal{V}_t(\xi)) = \sum_{[\gamma_0;\gamma,\delta]} c(\gamma_0) \cdot \mathrm{vol}(I(\mathbb{Q})\backslash I(\mathbb{A}_f)) \cdot O_\gamma(f^p) \cdot TO_\delta(\phi_r) \cdot \mathrm{Tr}(\gamma_0) \tag{0.4}$$

where the sum is over a set of representatives for the equivalence classes of effective triples and $c(\gamma_0)$ is the number of times the equivalence class of $(\gamma_0;\gamma,\delta)$ arises from a pair (φ,ε). This differs from [Kottwitz 1990, 3.1] only in the description of the index set. In §7 we show that $(\gamma_0;\gamma,\delta)$ is effective if and only if its Kottwitz invariant $\alpha(\gamma_0;\gamma,\delta)$ is defined and equals 1, and so we obtain formula (3.1) of [Kottwitz 1990]:

$$\textstyle\sum_{t'} \mathrm{Tr}(\mathcal{T}(g)^{(r)}|\mathcal{V}_t(\xi)) = \sum_{\alpha(\gamma_0;\gamma,\delta)=1} c(\gamma_0) \cdot \mathrm{vol}(I(\mathbb{Q})\backslash I(\mathbb{A}_f)) \cdot O_\gamma(f^p) \cdot TO_\delta(\phi_r) \cdot \mathrm{Tr}(\gamma_0). \tag{0.5}$$

Although the derivation of (0.5) from (0.1) is not contained in [Langlands and Rapoport 1987], they do prove most of the results required for it.

In §8 we briefly review results of [Kottwitz 1990] concerning the stabilization of (0.1).

In three appendices we provide background material for the rest of the article.

The notion of a groupoid (in schemes) is a natural generalization of that of a group scheme, and affine groupoids classify general Tannakian categories in exactly the same way that affine group schemes classify neutral Tannakian categories. In Appendix A we review the theory of groupoids, and we state the main theorems of Tannakian categories.

Throughout the paper, we have used results of Kottwitz (extending earlier results of Tate and Langlands) concerning the cohomology of reductive groups. In proving his results, Kottwitz used Langlands's theory of the dual group, but recently Borovoi has shown that it is possible to give a more direct derivation of slightly stronger results by using the cohomology of "crossed modules". In Appendix B we derive the results from this point of view.

Finally in Appendix C, we explain the relation of the problem of computing the zeta function of a local system $\mathcal{V}$ on a Shimura variety (defined in terms of intersection cohomology) to finding $\sum_{t'} \mathrm{Tr}(\mathcal{T}(g)^{(r)}|\mathcal{V}_t(\xi))$.

History. Although important work had been done earlier in special cases by Eichler, Kuga, Ihara, Shimura, and others, Langlands was the first to attempt to understand the zeta function of a Shimura variety in all generality. In his Jugentraum paper [Langlands 1976], Langlands stated a conjecture describing $\mathrm{Sh}_p(\mathbb{F})$. While this was a crucial first step, the conjecture did not succeed in describing the isomorphism class of the triple $(\mathrm{Sh}_p(\mathbb{F}), \Phi, \times)$—roughly speaking, it grouped together terms on the right hand side of (0.1) corresponding to locally isomorphic φ's—and, in fact, was too imprecise to permit passage to (0.5). Moreover, it was based on the study of examples rather than a heuristic understanding of the general case, which is perhaps why it required successive corrections [Langlands 1977, p1299; Langlands 1979, p1173]. The conjecture of Langlands and Rapoport removes both these defects: it does give a precise description of the isomorphism class of $(\mathrm{Sh}_p(\mathbb{F}), \Phi, \times)$ and, as is demonstrated in this article, it is sufficiently strong to imply (0.5); moreover, as we noted above, the general form of the conjecture is suggested by Deligne's philosophy that Shimura varieties with rational weight should be moduli varieties for motives.[2]

[2]This philosophy is now a theorem for Shimura varieties of abelian type (this class excludes only those defined by groups containing factors of type E_6, E_7, or certain groups of type D); see [Milne 1991c], which also shows that, for Shimura varieties of abelian type, the conjecture of Langlands and Rapoport is a consequence of other standard conjectures. I should also mention that another very important motivation for Langlands and Rapoport was their desire to include

In 1974 Langlands sent Rapoport a long letter outlining a proof of his Jugendtraum conjecture for certain Shimura varieties of PEL-type—since these varieties are moduli schemes for polarized abelian varieties with endomorphism and level structure, one has a description of $\mathrm{Sh}_p(\mathbb{F})$ in terms of the isomorphism classes of such systems over $\mathbb{F}$. At the time of the Corvallis conference (1977) it was believed that the outline could be completed to a proof, but this turned out to be impossible, and the conjecture was not proved, even in the case of the Siegel modular variety, for more than ten years.[3]

In the intervening period, Zink obtained a number of partial results. For example, he proves in [Zink 1983] that, for a Shimura variety of PEL-type, every isogeny class in the family parametrized by the variety over $\mathbb{F}$ lifts to an object in characteristic zero that is in the family and of CM-type.

In his paper [Langlands 1979], Langlands assumed his Jugendtraum conjecture, and showed that the zeta function of a Shimura variety defined by a quaternion algebra over a totally real field is an alternating product of automorphic L-functions.

In a series of papers, Kottwitz made far-reaching extensions of the work of Langlands. Starting from Langlands's original conjecture, he was led to a conjecture for the order of $\mathrm{Sh}_p(\mathbb{F}_q)$ of the same general shape as (0.5) [Kottwitz 1984b]. Later, he introduced the Kottwitz invariant $\alpha(\gamma_0;\gamma,\delta)$ of a triple, which, in his talk at the 1988 Ann Arbor conference [Kottwitz 1990, 3.1], allowed him to formulate his conjecture (0.5). In the same talk he showed that, if one assumed some standard conjectures in the theory of automorphic representations, most notably the fundamental lemma, then it was possible to stabilize (0.5), i.e., put it in a form more appropriate for comparison with the terms arising (via the trace formula) from the zeta function (see §8 below). Finally he proved that (0.5) holds for Siegel modular varieties.

One difficulty Kottwitz had to overcome in proving (0.5) for Siegel modular varieties was that of giving an explicit description of the polarized Dieudonné module of the reduction modulo p of an abelian variety of CM-type—he needed this to show that the triple $(\gamma_0;\gamma,\delta)$ arising from a polarized abelian variety over $\mathbb{F}$ has Kottwitz invariant 1, and therefore contributes to the right hand side of (0.5) (cf. [Kottwitz 1990, 12.1]). Later Wintenberger was to clarify this result by obtaining

the case of mild bad reduction; this aspect of their work will be ignored throughout the article.

[3]The author, then a novice in the field of Shimura varieties, had the misfortune to be asked to explain to the Corvallis conference Langlands's letter in the case of a Shimura variety defined by a totally indefinite quaternion algebra over a totally real field. Happily, for reasons to do with the nonexistence of L-indistinguishability, this case is easier than that of a general PEL-variety, and the articles [Milne 1979a; 1979b] contain a complete proof of Langlands's conjecture for this case, albeit one based more on the theorem of Tate and Honda than on Langlands's letter.

a similar statement for a CM-motive and all its Hodge cycles [Wintenberger 1991], and the statement can now be regarded as a rather immediate consequence of the theory of Fontaine.[4]

Kottwitz's proof of (0.5) did not suggest a proof of conjecture (0.1) of Langlands and Rapoport, and in fact it was not until two years later that this stronger result was obtained [Milne 1991a]. In their paper, Langlands and Rapoport had proved (0.1) for the Siegel modular variety, but only under the assumption of Grothendieck's standard conjectures and the Tate conjecture for varieties over finite fields, and the Hodge conjecture for abelian varieties. The main idea in [Milne 1991a] is to construct explicitly a groupoid from a polarized abelian variety over $\mathbb{F}$ using only the polarization and the endomorphisms of the variety (rather than all its algebraic cycles), and to show that this is a sufficiently fine object to obtain a proof of (0.1) without assumptions.

Comments. One may ask why we should bother with (0.1) since (0.5) is all one needs for the zeta function. The simplest answer is that (0.1) is the stronger result, and hence the more challenging problem, but there are more intelligent responses.

First, the definition of the canonical model of a Shimura variety is indirect. In particular, it provides no description of the points of the variety with coordinates in the fields containing the reflex field, and in general we have no such description. From the point of view of the geometry of the Shimura variety, (0.1) gives a remarkably precise description of the points of the Shimura variety in finite fields, and it suggests a similar description for the points in any local field containing the reflex field. The formula (0.5) has no such direct geometric significance, and suffers from the same defect as Langlands's original conjecture in being a sum over locally isomorphic objects whereas one wants a finer sum over globally isomorphic objects.

Second, in the theory of Shimura varieties one typically proves a statement for some (small) class of Shimura varieties, and extends it to a larger class by the intermediary of connected Shimura varieties. Formula (0.5) seems to be badly adapted for this approach whereas it should work well for (0.1).

Third, as was mentioned above, it has been proved that, for Shimura varieties of abelian type whose weight is defined over $\mathbb{Q}$, (0.1) follows from other standard conjectures. For me, this is the most compelling evidence for (0.1), and, when combined with the results of this article, also the most compelling evidence for (0.5).

[4]However, the finer result of [Reimann and Zink 1988] has not yet been obtained from this point of view.

In one change from [Langlands and Rapoport 1987][5], I use throughout the language of "groupoids" rather than "Galois-gerbs". In the author's (not so humble) opinion, the first is the correct notion, and the second should be expunged from the mathematical literature.

Throughout the paper I have assumed that the weight of the Shimura variety is defined over $\mathbb{Q}$. Presumably everything holds *mutatis mutandis* without this assumption if one replaces the pseudomotivic groupoid with the quasimotivic groupoid, but I haven't checked this. Also, as mentioned above, unlike Langlands and Rapoport, I have confined myself to the case of good reduction.

This article is largely expository; the notes at the end of each section give information on sources. My main purpose in writing it has been to make the beautiful ideas in [Langlands and Rapoport 1987] more easily accessible and to explain their relation to the (equally beautiful) ideas in [Kottwitz 1990].

Notations. Reductive groups are always connected. For such a group G, G^{der} denotes the derived group of G, $Z(G)$ denotes the centre of G, $G^{\mathrm{ad}} =_{df} G/Z(G)$ denotes the adjoint group of G, and $G^{\mathrm{ab}} =_{df} G/G^{\mathrm{der}}$ is the maximal abelian quotient of G. For any finite field extension $k \supset k_0$ and reductive group G over k, $\mathrm{Res}_{k/k_0} G$ denotes the group scheme over k_0 obtained from G by restriction of scalars. We write $(\mathbb{G}_m)_{k/k_0}$ for $\mathrm{Res}_{k/k_0} \mathbb{G}_m$. Also

$$\mathrm{Ker}^i(\mathbb{Q}, G) = \mathrm{Ker}(H^i(\mathbb{Q}, G) \to \Pi_\ell H^i(\mathbb{Q}_\ell, G))$$

(product over all primes ℓ of $\mathbb{Q}$, including p and ∞).

The expression (G, X) always denotes a pair defining a Shimura variety (see §1). We usually denote the corresponding reflex field $E(G, X)$ by E, and v is a fixed prime of E dividing a rational prime p and unramified over p.

We denote by $\mathbb{Q}^{\mathrm{al}}$ the algebraic closure of $\mathbb{Q}$ in $\mathbb{C}$, and by $\mathbb{Q}_p^{\mathrm{al}}$ an algebraic closure of E_v; $\mathbb{Q}_p^{\mathrm{un}}$ is the maximal unramified extension of $\mathbb{Q}_p$ contained in $\mathbb{Q}_p^{\mathrm{al}}$. The residue field of $\mathbb{Q}_p^{\mathrm{un}} \subset \mathbb{Q}_p^{\mathrm{al}}$ is denoted by $\mathbb{F}$, and $\kappa(v) \subset \mathbb{F}$ is the residue field of E at the prime v. We fix an extension of $E \hookrightarrow E_v$ to an embedding $\mathbb{Q}^{\mathrm{al}} \hookrightarrow \mathbb{Q}_p^{\mathrm{al}}$. We often use the following numbering:

$$m = [E_v \colon \mathbb{Q}_p] = [\kappa(v) \colon \mathbb{F}_p], \quad r = [\mathbb{F}_q \colon \kappa(v)], \quad n = mr = [\mathbb{F}_q \colon \mathbb{F}_p].$$

[5]The reader of [Langlands and Rapoport 1987] should be aware that throughout, including in the title, they use "gerb" where they should use "Galois-gerb"; the two concepts are not the same, and they should not be confused. In particular, the authors do not determine the gerb conjecturally attached to the category of motives over $\mathbb{F}$.

For a perfect field k, $W(k)$ is the ring of Witt vectors of k and $B(k)$ is the field of fractions of $W(k)$. When $k = \mathbb{F}$, we usually drop it from the notation. With the conventions of the last paragraph, we have $B(\mathbb{F}) \supset \mathbb{Q}_p^{\mathrm{un}}$ and $E_v = B(k(v))$.

The Frobenius element $x \mapsto x^p$ of $\mathrm{Gal}(\mathbb{F}/\mathbb{F}_p)$ is denoted by σ. We also use σ to denote the elements corresponding to σ under the canonical isomorphisms

$$\mathrm{Gal}(\mathbb{F}/\mathbb{F}_p) \approx \mathrm{Gal}(\mathbb{Q}_p^{\mathrm{un}}/\mathbb{Q}_p)) \approx \mathrm{Gal}(B(\mathbb{F})/B(\mathbb{F}_p)).$$

The ring of finite adèles $\widehat{\mathbb{Z}} \otimes \mathbb{Q}$ is denoted by $\mathbb{A}_f$, and the ring of finite adèles with the p-component omitted is denoted by $\mathbb{A}_f^p$; thus $\mathbb{A}_f = \mathbb{A}_f^p \times \mathbb{Q}_p$.

The Artin reciprocity maps of local and global class field theory are normalized so that a uniformizing parameter is mapped to the geometric Frobenius element. Thus if $\chi_{\mathrm{cyc}} \colon \mathrm{Gal}(\mathbb{Q}^{\mathrm{al}}/\mathbb{Q}) \to \widehat{\mathbb{Z}}^\times$ is the cyclotomic character, so that $\tau\zeta = \zeta^{\chi_{\mathrm{cyc}}(\tau)}$ for ζ a root of unity in $\mathbb{Q}^{\mathrm{al}}$, then $\mathrm{rec}_{\mathbb{Q}}(\chi_{\mathrm{cyc}}(\tau)) = \tau|\mathbb{Q}^{\mathrm{ab}}$.

Complex conjugation on $\mathbb{C}$ (or a subfield) is denoted by $z \mapsto \bar{z}$ or by ι. We often write $[*]$ for the equivalence class of $*$ or $(*)$, and we often use $=$ to denote a canonical isomorphism.

Finally we note that in §1 we correct a fundamental sign error in [Deligne 1979—see (1.10)]. Thus our signs will differ from papers using Deligne's paper as their reference.

1. SHIMURA VARIETIES

We review some of the theory of Shimura varieties.

The torus $\mathbb{S}$. We write $\mathbb{S}$ for the torus $(\mathbb{G}_m)_{\mathbb{C}/\mathbb{R}}$ over $\mathbb{R}$; thus

$$\mathbb{S}(\mathbb{R}) = \mathbb{C}^\times, \qquad \mathbb{S}(\mathbb{C}) = \mathbb{C}^\times \times \mathbb{C}^\times.$$

The last identification is made in such a way that the map $\mathbb{C}^\times \hookrightarrow \mathbb{C}^\times \times \mathbb{C}^\times$ induced by $\mathbb{R} \hookrightarrow \mathbb{C}$ is $z \mapsto (z, \bar{z})$. Let G be an algebraic group over $\mathbb{R}$. With any homomorphism $h \colon \mathbb{S} \to G$ there are associated homomorphisms,

$$\mu_h \colon \mathbb{G}_m \to G_{\mathbb{C}}, \quad z \mapsto h_{\mathbb{C}}(z, 1), \quad z \in \mathbb{G}_m(\mathbb{C}) = \mathbb{C}^\times,$$

and

$$w_h \colon \mathbb{G}_m \to G, \quad r \mapsto h(r)^{-1}, \quad r \in \mathbb{G}_m(\mathbb{R}) = \mathbb{R}^\times \subset \mathbb{C}^\times = \mathbb{S}(\mathbb{R}),$$

(the ***weight homomorphism***). To give a Hodge structure on a real vector space V is the same as to give a homomorphism $h \colon \mathbb{S} \to \mathrm{GL}(V)$: by convention, $h(z)$ acts on $V^{p,q}$ as multiplication by $z^{-p}\bar{z}^{-q}$.

Definition of a Shimura variety. The datum needed to define a Shimura variety is a pair (G, X) comprising a reductive group G over $\mathbb{Q}$ and a $G(\mathbb{R})$-conjugacy class X of homomorphisms $\mathbb{S} \to G_{\mathbb{R}}$ satisfying the following conditions:

(SV1) for each $h \in X$, the Hodge structure on the Lie algebra $\mathfrak{g}$ of G defined by $\mathrm{Ad} \circ h \colon \mathbb{S} \to \mathrm{GL}(\mathfrak{g}_{\mathbb{R}})$ is of type $\{(1,-1),(0,0),(-1,1)\}$;

(SV2) for each $h \in X$, $\mathrm{ad}\, h(i)$ is a Cartan involution on $G_{\mathbb{R}}^{\mathrm{ad}}$;

(SV3) G^{ad} has no factor defined over $\mathbb{Q}$ whose real points form a compact group;

(SV4) G^{ab} splits over a CM-field.

We write h_x, μ_x, and w_x for the homomorphisms corresponding to a point $x \in X$; thus $h_{gx} = \mathrm{ad}\, g \circ h_x$ for $g \in G(\mathbb{R})$.

The set X has a canonical $G(\mathbb{R})$-invariant complex structure for which the connected components are symmetric Hermitian domains.

For each compact open subgroup K of $G(\mathbb{A}_f)$,

$$\mathrm{Sh}_K(G, X) =_{df} G(\mathbb{Q})\backslash X \times G(\mathbb{A}_f)/K$$

is a finite disjoint union of quotients of X by arithmetic subgroups. According to Baily and Borel (1966), this space has a natural structure of a (nonconnected) quasi-projective variety over $\mathbb{C}$. The ***Shimura variety*** $\mathrm{Sh}(G, X)$ is the projective system of these varieties, or (what amounts to the same thing) the limit of the system, together with the action of $G(\mathbb{A}_f)$ defined by the rule:

$$[x, a] \cdot g = [x, ag], \quad x \in X, \quad a, g \in G(\mathbb{A}_f).$$

The set of complex points of $\mathrm{Sh}(G, X)$ is

$$\mathrm{Sh}(G, X)(\mathbb{C}) = G(\mathbb{Q})\backslash X \times G(\mathbb{A}_f)/Z(\mathbb{Q})^-,$$

where $Z = Z(G)$ and $Z(\mathbb{Q})^-$ is the closure of $Z(\mathbb{Q})$ in $Z(\mathbb{A}_f)$ (see Deligne 1979, 2.1). In forming the quotient, $G(\mathbb{Q})$ acts on the left on X and $G(\mathbb{A}_f)$, and $Z(\mathbb{Q})^-$ acts only on $G(\mathbb{A}_f)$. Because of (SV4), the largest split subtorus of $Z_{\mathbb{R}}$ is defined over $\mathbb{Q}$; if it is split over $\mathbb{Q}$, then $Z(\mathbb{Q})$ is closed in $Z(\mathbb{A}_f)$, and

$$\mathrm{Sh}(G, X)(\mathbb{C}) = G(\mathbb{Q})\backslash X \times G(\mathbb{A}_f).$$

Axiom (SV1) implies that the Hodge structure on $\mathfrak{g}$ defined by $\mathrm{Ad} \circ h_x$ has weight zero for each $x \in X$. Therefore $w_x(\mathbb{G}_m) \subset Z(G)$, and w_x is independent of x—we write it w_X, and refer to it as the ***weight*** of the Shimura variety.

Examples of Shimura varieties. We list some Shimura varieties of interest.

Example 1.1. Let L be a finite-dimensional semisimple algebra over $\mathbb{Q}$ with an involution $*$, and let V be a finite-dimensional vector space over $\mathbb{Q}$ endowed with the structure of a faithful L-module and a nondegenerate skew-symmetric form ψ such that

$$\psi(ax, y) = \psi(x, a^* y), \quad \text{all } a \in L, \ x, y \in V.$$

Let E be the centre of L, and let F be the subalgebra of E of elements fixed by $*$. Fix a torus C such that

$$\mathbb{G}_m \subset C \subset (\mathbb{G}_m)_{F/\mathbb{Q}},$$

and define G to be the identity component of the group G' of symplectic C-similitudes:

$$\begin{aligned} G'(\mathbb{Q}) &= \{a \in \mathrm{GL}_L(V) \mid \psi(ax, ay) = \psi(\nu(a)x, y), \text{ some } \nu(a) \in C(\mathbb{Q})\}, \\ &= \{a \in \mathrm{GL}_L(V) \mid a^* \cdot a \in C(\mathbb{Q})\}. \end{aligned}$$

Then $a \mapsto \nu(a) = a^* \cdot a$ defines a homomorphism of algebraic groups $\nu \colon G \to C$. Assume there is a homomorphism $h_0 \colon \mathbb{S} \to G_{\mathbb{R}}$ such that the Hodge structure (V, h_0) is of type $\{(-1,0),(0,-1)\}$ and $2\pi i\psi$ is a polarization for (V, h_0). Then the involution $*$ is positive, and when we take X to be the set of $G(\mathbb{R})$-conjugates of h_0, the pair (G, X) satisfies the axioms for a Shimura variety (see Deligne 1971a, 4.9). A Shimura variety arising in this way from an algebra with involution and a symplectic representation of the algebra is said to be of ***PEL-type***. Deligne (ibid. §5) gives a description of the groups G that occur in this way. For Shimura varieties of PEL-type, the weight is automatically defined over $\mathbb{Q}$.

Example 1.2 (Special case of (1.1)). As in (Gordon 1991, §6), let E be a CM-field of degree $2g$ over $\mathbb{Q}$ with F as its largest totally real subfield, let V be an E-vector space of dimension 3, and let J be an E-valued Hermitian form on V with signature $(2,1)$ at r infinite primes of F and signature $(3,0)$ at the remainder. Assume $r \geq 1$.

Take L in (1.1) to be E with complex conjugation as the involution. The subfield of E fixed by the involution is F. Regard V as a vector space over $\mathbb{Q}$, and take ψ to be the imaginary part of J. Finally take $C = (\mathbb{G}_m)_{F/\mathbb{Q}}$. For the obvious choice of h_0, the construction in (1.1) leads to the Shimura variety $\mathrm{Sh}(G, X)$ discussed in (Gordon 1991, §6). In the special case that $g = 1$, $\mathrm{Sh}(G, X)$ is the Picard modular surface (ibid. §1–§5). The derived group of G is simply connected and $G^{\mathrm{ab}} = (\mathbb{G}_m)_{E/\mathbb{Q}}$.

Example 1.3. Let F be a totally real number field, and let L be a quaternion algebra over F that splits at at least one real prime of F. Let G be the algebraic group $\mathrm{GL}_1(L)$ over $\mathbb{Q}$. Then $G_{\mathbb{R}} = \prod G_v$ where v runs over the real primes of F and G_v is isomorphic to $\mathrm{GL}_2(\mathbb{R})$ or $\mathrm{GL}_1(\mathbb{H})$ according as L does or does not split at v. Define h_0 to be the homomorphism $\mathbb{S} \to G_{\mathbb{R}}$ such that the projection of $h_0(a+bi)$ to $G_v(\mathbb{R})$ is $\left(\begin{smallmatrix} a & -b \\ b & a \end{smallmatrix}\right)$ or 1 in the two cases. When we take X to be the set of $G(\mathbb{R})$-conjugates of h_0, the pair (G, X) satisfies the axioms for a Shimura variety. The weight is defined over $\mathbb{Q}$ if and only if L splits at all real primes of F, in which case the Shimura variety is of PEL-type.

Example 1.4. (Special case of (1.1)) By a ***symplectic space*** over $\mathbb{Q}$, I mean a vector space V over $\mathbb{Q}$ together with a nondegenerate skew-symmetric form ψ on V. The group $\mathrm{GSp}(V, \psi)$ of ***symplectic similitudes*** has rational points,

$$\{a \in \mathrm{GL}(V) \mid \psi(ax, ay) = \nu(a)\psi(x, y), \text{ some } \nu(a) \in \mathbb{Q}^{\times}\}.$$

The ***Siegel double space*** S consists of all rational Hodge structures on V of type $\{(-1,0),(0,-1)\}$ for which $\pm 2\pi i\,\psi$ is a polarization. It is a $\mathrm{GSp}(\mathbb{R})$-conjugacy class of maps $\mathbb{S} \to \mathrm{GSp}_{\mathbb{R}}$, and the Shimura variety $\mathrm{Sh}(\mathrm{GSp}, S)$ is called the ***Siegel modular variety*** (see [Deligne 1971b, 1.6; 1979, 1.3.1]).

Example 1.5. A Shimura variety $\mathrm{Sh}(G, X)$ is said to be of ***Hodge type*** if there is a symplectic space (V, ψ) and an injective homomorphism $G \hookrightarrow \mathrm{GSp}(V, \psi)$ carrying X into S. Thus a Siegel modular variety is of Hodge type, and a Shimura variety of PEL-type is of Hodge type if the group C in its definition is taken to be $\mathbb{G}_m$.

A Shimura variety $\mathrm{Sh}(G, X)$ is of Hodge type if and only if the following conditions hold (see [Deligne 1979, 2.3.2]):

(i) the weight is defined over $\mathbb{Q}$;

(ii) $w_X(\mathbb{G}_m)$ is the only split subtorus of $Z(G)_{\mathbb{R}}$;

(iii) there is a faithful representation $\xi\colon G \hookrightarrow \mathrm{GL}(V)$ of G such that $(V, \xi \circ h_x)$ is of type $\{(-1,0),(0,-1)\}$ for all $x \in X$.

Example 1.6. Let T be a torus over $\mathbb{Q}$ that is split by a CM-field. For any homomorphism $h\colon \mathbb{S} \to T_{\mathbb{R}}$, $(T, \{h\})$ satisfies the conditions (SV), and so defines a Shimura variety. Its points are

$$\mathrm{Sh}(T, \{h\}) = T(\mathbb{A}_f)/T(\mathbb{Q})^{-}.$$

The reflex field. For any algebraic group G over a field k, and any field k' containing k, write $\mathcal{C}(k')$ for the set of $G(k')$-conjugacy classes of homomorphisms $\mathbb{G}_m \to G_{k'}$:

$$\mathcal{C}(k') = \mathrm{Hom}(\mathbb{G}_m, G_{k'})/G(k').$$

Note that a map $k' \to k''$ defines a map $\mathcal{C}(k') \to \mathcal{C}(k'')$; in particular, when k' is Galois over k, $\mathrm{Gal}(k'/k)$ acts on $\mathcal{C}(k')$.

Proposition 1.7. *Let G be a reductive group over a field k of characteristic zero.*

(a) *For any maximal k-split torus S in G_k, with k-Weyl group Ω, the map $X_*(S)/\Omega \to \mathcal{C}(k)$ is bijective.*
(b) *If G is quasi-split over k, then $\mathcal{C}(k) = \mathcal{C}(k^{\mathrm{al}})^{\mathrm{Gal}(k^{\mathrm{al}}/k)}$.*
(c) *If G is split over k (for example, if k is algebraically closed), then the map $\mathcal{C}(k) \to \mathcal{C}(k')$ is a bijection for any $k' \supset k$.*

Proof. The first two statements are proved in [Kottwitz 1984b, 1.1.3]—the hypothesis there that G^{der} is simply connected is not used in the proof of (a) or (b), and the hypothesis that G is quasi-split is not used in the proof of (a). The remaining statement follows (a). □

Now consider a Shimura variety $\mathrm{Sh}(G, X)$, and let $\mathfrak{c}(X)$ be the $G(\mathbb{C})$-conjugacy class of homomorphisms $\mathbb{G}_m \to G_{\mathbb{C}}$ containing μ_x for $x \in X$. According to (c) of the proposition, $\mathfrak{c}(X)$ corresponds to an element $\mathfrak{c}(X)_{\mathbb{Q}^{\mathrm{al}}}$ of $\mathcal{C}(\mathbb{Q}^{\mathrm{al}})$. The Galois group $\mathrm{Gal}(\mathbb{Q}^{\mathrm{al}}/\mathbb{Q})$ acts on $\mathcal{C}(\mathbb{Q}^{\mathrm{al}})$, and the subfield of $\mathbb{Q}^{\mathrm{al}}$ corresponding to the stabilizer of $\mathfrak{c}(X)_{\mathbb{Q}^{\mathrm{al}}}$ is defined to be the ***reflex field*** $E(G, X)$ of $\mathrm{Sh}(G, X)$. Thus $\tau \in \mathrm{Gal}(\mathbb{Q}^{\mathrm{al}}/\mathbb{Q})$ fixes $E(G, X)$ if and only if it fixes $\mathfrak{c}(X)_{\mathbb{Q}^{\mathrm{al}}}$, and this condition characterizes $E(G, X)$.

The reciprocity map. Consider a pair (T, x) as in (1.6). The reflex field $E =_{df} E(T, x)$ is the field of definition of the cocharacter μ_x of T. On applying $\mathrm{Res}_{E/\mathbb{Q}}$ to the homomorphism $\mu_x \colon \mathbb{G}_{mE} \to T_E$, and composing with the norm map, we obtain a homomorphism

$$N_x \colon \mathrm{Res}_{E/\mathbb{Q}}\mathbb{G}_{mE} \xrightarrow{\mathrm{Res}_{E/\mathbb{Q}}\mu_x} \mathrm{Res}_{E/\mathbb{Q}}T_E \xrightarrow{\mathrm{Norm}_{E/\mathbb{Q}}} T.$$

For any $\mathbb{Q}$-algebra R, this gives a homomorphism

$$N_x \colon (E \otimes R)^\times \to T(R).$$

Let $T(\mathbb{Q})^{-}$ be the closure of $T(\mathbb{Q})$ in $T(\mathbb{A}_f)$. The ***reciprocity map***

$$r(T,x)\colon \mathrm{Gal}(E^{\mathrm{ab}}/E) \to T(\mathbb{A}_f)/T(\mathbb{Q})^{-}$$

is defined as follows: let $\tau \in \mathrm{Gal}(E^{\mathrm{ab}}/E)$, and let $s \in \mathbb{A}_E^{\times}$ be such that $\mathrm{rec}_E(s) = \tau$; write $s = s_\infty \cdot s_f$ with $s_\infty \in (E \otimes \mathbb{R})^{\times}$ and $s_f \in (E \otimes \mathbb{A}_f)^{\times}$; then $r(T,x)(\tau) = N_x(s_f)$ (mod $T(\mathbb{Q})^{-}$).

The canonical model of $\mathrm{Sh}(G,X)$ **over** $E(G,X)$. A ***special pair*** (T,x) in (G,X) is a torus $T \subset G$ together with a point x of X such that h_x factors through $T_{\mathbb{R}}$. Clearly $E(T,x) \supset E(G,X)$.

By a ***model*** of $\mathrm{Sh}(G,X)$ over a subfield k of $\mathbb{C}$, we mean a scheme S over k endowed with an action of $G(\mathbb{A}_f)$ (defined over k) and a $G(\mathbb{A}_f)$-equivariant isomorphism $\mathrm{Sh}(G,X) \to S \otimes_k \mathbb{C}$. We use this isomorphism to identify $\mathrm{Sh}(G,X)(\mathbb{C})$ with $S(\mathbb{C})$.

Theorem 1.8. *There exists a model of $\mathrm{Sh}(G,X)$ over $E(G,X)$ with the following property: for all special pairs $(T,x) \subset (G,X)$ and elements $a \in G(\mathbb{A}_f)$, the point $[x,a]$ is rational over $E(T,x)^{\mathrm{ab}}$ and $\tau \in \mathrm{Gal}(E(T,x)^{\mathrm{ab}}/E(T,x))$ acts on $[x,a]$ according to the rule*

$$\tau[x,a] = [x, a \cdot r(\tau)], \text{ where } r = r(T,x).$$

The model is uniquely determined by this condition up to a unique isomorphism.

Proof. The uniqueness is proved in [Deligne 1971b]. For most Shimura varieties, the existence is proved in [Deligne 1979], and for the remainder it is proved in [Milne 1983]. □

The model determined by the theorem is called the ***canonical model*** of $\mathrm{Sh}(G,X)$. We now use $\mathrm{Sh}(G,X)$ to denote the canonical model of the Shimura variety over $E(G,X)$, or its base change to any field $k \supset E(G,X)$.

Example 1.9. Let $h\colon \mathbb{S} \to \mathbb{G}_{m\mathbb{R}}$ be the map $z \mapsto z\bar{z}$. The Shimura variety $\mathrm{Sh}(\mathbb{G}_m, \{h\})$ has complex points

$$\mathrm{Sh}(\mathbb{G}_m, \{h\})(\mathbb{C}) = \mathbb{Q}^{\times} \backslash \mathbb{A}_f^{\times}.$$

The reflex field is the field of definition of $\mu_h = \mathrm{id}\colon \mathbb{G}_m \to \mathbb{G}_m$, which is $\mathbb{Q}$. The reciprocity map $r = r(\mathbb{G}_m, h)\colon \mathrm{Gal}(\mathbb{Q}^{\mathrm{ab}}/\mathbb{Q}) \to \mathbb{A}_f^{\times}/\mathbb{Q}^{\times}$ can be described as follows: let $\tau \in \mathrm{Gal}(\mathbb{Q}^{\mathrm{ab}}/\mathbb{Q})$, and let $s \in \mathbb{A}^{\times}$ be such that $\mathrm{rec}_{\mathbb{Q}}(s) = \tau$; then $r(\tau) = s_f$. The canonical model of $\mathrm{Sh}(\mathbb{G}_m, \{h\})$ is the (unique) scheme S of dimension zero over $\mathbb{Q}$

such that, for any algebraically closed field k containing $\mathbb{Q}$, $S(k) = \mathbb{Q}^{\times}\backslash\mathbb{A}_f^{\times}$ with $\tau \in \mathrm{Gal}(k/\mathbb{Q})$ acting as multiplication by $r(\tau|\mathbb{Q}^{\mathrm{ab}})$. Here $\mathbb{Q}^{\mathrm{ab}}$ denotes the largest abelian extension of $\mathbb{Q}$ contained in k; the expression $r(\tau|\mathbb{Q}^{\mathrm{ab}})$ makes sense because $\mathrm{Gal}(\mathbb{Q}^{\mathrm{ab}}/\mathbb{Q})$ is independent of k.

We verify that this is the moduli variety for level structures on the Tate motive $\mathbb{Q}(1)$. Let k be an algebraic extension of $\mathbb{Q}$, and let k^{al} be an algebraic closure of k. Define

$$\mathbb{A}_f(1) = (\varprojlim \mu_n(k^{\mathrm{al}})) \otimes_{\mathbb{Z}} \mathbb{Q}$$

(étale realization of the Tate motive). It is a free $\mathbb{A}_f$-module of rank 1, and $\mathrm{Gal}(k^{\mathrm{al}}/k)$ acts on it through its action on the roots of unity. More explicitly, $\tau \in \mathrm{Gal}(k^{\mathrm{al}}/k)$ acts on $\mathbb{A}_f(1)$ as multiplication by $\chi_{\mathrm{cyc}}(\tau)$, where χ_{cyc} is the cyclotomic character (see Notations).

A ***level structure*** on $\mathbb{Q}(1)$ defined over k is an isomorphism $\eta\colon \mathbb{A}_f \to \mathbb{A}_f(1)$ of $\mathrm{Gal}(k^{\mathrm{al}}/k)$-modules. Since $\mathrm{Gal}(k^{\mathrm{al}}/k)$ acts trivially on $\mathbb{A}_f$, a level structure can exist only when k contains all roots of 1, i.e., when $k \supset \mathbb{Q}^{\mathrm{ab}}$. An ***isomorphism of level structures*** $a\colon \eta \to \eta'$ is an automorphism a of $\mathbb{Q}(1)$ (element of $\mathbb{Q}^{\times}$) such that $\eta' = a \circ \eta$. Let $\mathcal{M}(k)$ be the set of isomorphism classes of level structures on $\mathbb{Q}(1)$ defined over k. When k is algebraically closed, there is a canonical isomorphism $\mathcal{M}(k) \to \mathrm{Sh}(\mathbb{G}_m, \{h\})(k)$: choose an isomorphism $\beta\colon \mathbb{Q}(1)_B \to \mathbb{Q}$ where $\mathbb{Q}(1)_B$ denotes the vector space $2\pi i\mathbb{Q}$ (Betti realization of $\mathbb{Q}(1)$); on tensoring this with $\mathbb{A}_f$ we obtain an isomorphism $\beta \otimes 1\colon \mathbb{A}_f(1) = \mathbb{A}_f \otimes \mathbb{Q}(1)_B \to \mathbb{A}_f$. For any level structure η, $(\beta \otimes 1) \circ \eta$ is an automorphism of $\mathbb{A}_f$, i.e., an element of $\mathbb{A}_f^{\times}$. The class of this element in $\mathbb{Q}^{\times}\backslash\mathbb{A}_f^{\times}$ is independent of the choice of β, and depends only on the isomorphism class of η. Thus we have a canonical bijection:

$$\mathcal{M}(k) \to \mathbb{Q}^{\times}\backslash\mathbb{A}_f^{\times} = \mathrm{Sh}(\mathbb{G}_m, \{h\})(k).$$

The fact that this commutes with the actions of $\mathrm{Gal}(k^{\mathrm{al}}/\mathbb{Q})$ comes down to the formula recalled in the Notations:

$$\mathrm{rec}_{\mathbb{Q}}(\chi_{\mathrm{cyc}}(\tau)) = \tau|\mathbb{Q}^{\mathrm{ab}}.$$

Remark 1.10. The above definition of the reciprocity map $r(T, x)$ is correct. There is a sign error in the definition in [Deligne 1979], 2.2.3, which is repeated in all subsequent papers using that paper as reference including, alas, [Milne 1990]. (To verify that the sign is correct, it is necessary to trace through the signs in the theory of complex multiplication, but the above example provides rather convincing evidence of its correctness.)

Hyperspecial subgroups. We review part of the theory of hyperspecial subgroups that will be needed later in the article.

To a reductive group G over a nonarchimedean local field F, Bruhat and Tits attach a building $\mathcal{B}(G, F)$ [Tits 1979]. This is, in particular, a set with a left action of $G(F)$. Certain points of $\mathcal{B}(G, F)$ are said to be ***hyperspecial*** (ibid. 1.10.2), and the stabilizer in $G(F)$ of such a point contains a maximal compact subgroup of $G(F)$, called the ***hyperspecial subgroup*** of $G(F)$ attached to the point. These subgroups can be characterized independently of the building as follows: a subgroup $K \subset G(F)$ is hyperspecial if and only if there is a smooth group scheme $G_{\mathcal{O}}$ over the ring of integers $\mathcal{O}$ in F whose generic fibre is G, whose special fibre is a connected reductive group over the residue field, and whose group of $\mathcal{O}$-valued points is K (ibid. 3.8.1). Hyperspecial subgroups exist in $G(F)$ if and only if G is ***unramified*** over F, i.e., is quasi-split over F and splits over an unramified extension of F (ibid. 1.10.2).

Let x be a hyperspecial point of $\mathcal{B}(G, F)$, and let $K = G_{\mathcal{O}}(\mathcal{O})$ be the corresponding hyperspecial group. If F' is an unramified extension of F, then there is a canonical map $\mathcal{B}(G, F) \to \mathcal{B}(G, F')$ which is equivariant relative to $G(F) \to G(F')$. Moreover, x maps to a hyperspecial point x' in $\mathcal{B}(G, F')$ whose stabilizer is $K' = G_{\mathcal{O}}(\mathcal{O}')$ where $\mathcal{O}'$ is the ring of integers in F'.

For an unramified extension F' of F, write $\mathcal{V}(F')$ for the $G(F')$-orbit of the image of x in $\mathcal{B}(G, F')$. Thus $\mathcal{V}(F') = (G(F')/G_{\mathcal{O}}(\mathcal{O}')) \cdot x'$. Those wishing to avoid thinking about the building can identify $\mathcal{V}(F')$ with $G(F')/G_{\mathcal{O}}(\mathcal{O}')$. Note that there is a surjection

$$\mathcal{V}(F') \times \mathcal{V}(F') \to G_{\mathcal{O}}(\mathcal{O}') \backslash G(F')/G_{\mathcal{O}}(\mathcal{O}'), \quad (gx, g'x) \mapsto [g'^{-1} \cdot g],$$

whose fibres are the orbits of $G(F')$ acting on $\mathcal{V}(F') \times \mathcal{V}(F')$.

By definition, the building $\mathcal{B}(G, F)$ is a union of apartments corresponding to the maximal F-split tori in G. Suppose now that G is split over F. If the hyperspecial point corresponding to $K = G_{\mathcal{O}}(\mathcal{O})$ lies in the apartment corresponding to the torus S, then K contains a set of representatives for the Weyl group Ω of S, and there is a decomposition:

$$G(F) = K \cdot S(F) \cdot K,$$

(Cartan decomposition, ibid. 3.3.3). Moreover, there is a bijection

$$X_*(S)/\Omega \to K \backslash G(F)/K, \quad \mu \mapsto [\mu(p)].$$

On combining this with the bijection (1.7a)

$$X_*(S)/\Omega \to \mathcal{C}(F) \quad (G(F)\text{-conjugacy classes of cocharacters of } \mathrm{G}_F)$$

and the above surjection, we obtain a canonical map

$$\mathrm{inv}\colon \mathcal{V}(F) \times \mathcal{V}(F) \to \mathcal{C}(F)$$

whose fibres are the orbits of $G(F)$. It has the following description: $\mathrm{inv}(gx, g'x) = [\mu]$ if μ factors through S and

$$[g'^{-1}g] = [\mu(p)] \text{ in } K\backslash G(F)/K.$$

(See also [Langlands and Rapoport 1987, p168] and [Kottwitz 1984b, 1.3.3.]

Notes. There is no satisfactory detailed exposition of the theory of Shimura varieties. Brief accounts can be found in [Deligne 1971a; 1971b; 1979], and [Milne 1990]. For more details on buildings, see [Tits 1979].

2. INTEGRAL CANONICAL MODELS

In the last section, we explained the notion of a canonical model of a Shimura variety defined over its reflex field. Here we introduce the notion of an ***integral*** canonical model defined over the ring of integers in a completion of the reflex field, without which the problem of describing the points modulo p is not well-posed.[6]

Definitions. Fix a prime number p and a prime v of $E(G,X)$ lying over it, and write E_v for the completion of $E(G,X)$ at v. Fix also a compact open subgroup K_p of $G(\mathbb{Q}_p)$, and let

$$\mathrm{Sh}_p(G,X) = \mathrm{Sh}(G,X)/K_p.$$

It is a scheme over $E(G,X)$ with a continuous action of $G(\mathbb{A}_f^p)$. Let L be a finite extension of E_v, and let $\mathcal{O}_L$ be the ring of integers in L.

Definition 2.1. A ***model*** of $\mathrm{Sh}_p(G,X)$ over $\mathcal{O}_L$ is a scheme S over $\mathcal{O}_L$ together with a continuous action of $G(\mathbb{A}_f^p)$ and a $G(\mathbb{A}_f^p)$-equivariant isomorphism

$$\gamma\colon S \otimes_{\mathcal{O}_L} L \to \mathrm{Sh}_p(G,X)_L.$$

Recall [Deligne 1979, 2.7.1] that to say that S is a scheme over $\mathcal{O}_L$ with a ***continuous action*** of $G(\mathbb{A}_f^p)$ means that S is a projective system of schemes

[6]This lacuna has not, of course, prevented the problem being posed. It is the lack of a good notion of an integral canonical model in the case of bad reduction that has discouraged the author from considering that case.

(S_K) over $\mathcal{O}_L$ indexed by the compact open subgroups K of $G(\mathbb{A}_f^p)$, and that there is an action ρ of $G(\mathbb{A}_f^p)$ on the system defined by morphisms

$$\rho_K(g)\colon S_{gKg^{-1}} \to S_K.$$

Moreover $\rho_K(k)$ is the identity if $k \in K$. Therefore, for K' normal in K, the $\rho_{K'}(k)$ define an action of the finite group K/K' on $S_{K'}$, and it is required that $S_{K'}/(K/K') \xrightarrow{\approx} S_K$. The system is determined by its limit $S' = \varprojlim_K S_K$ together with the action of $G(\mathbb{A}_f^p)$ on S'; in fact $S_K = S'/K$. Therefore, we usually do not distinguish $S = (S_K)$ from its limit S'.

Definition 2.2. A model $S = (S_K)$ of $\mathrm{Sh}_p(G, X)$ over $\mathcal{O}_L$ is said to be ***smooth*** if there is a compact open subgroup K_0 of $G(\mathbb{A}_f^p)$ such that S_K is smooth over $\mathcal{O}_L$ for all $K \subset K_0$, and $S_{K'}$ is étale over S_K for all $K' \subset K \subset K_0$.

Remark 2.3. Assume that S_K is flat over $\mathcal{O}_L$ for all K; then (S_K) is smooth if and only if there is a compact open subgroup K_0 of $G(\mathbb{A}_f^p)$ such that the special fibre of $S_K \to \operatorname{Spec} \mathcal{O}_L$ is a smooth scheme over $\kappa(v)$ for all $K \subset K_0$, and the special fibre of $S_{K'} \to S_K$ is étale for all $K' \subset K \subset K_0$.

Proposition 2.4. *Let $S = \varprojlim S_K$ be a smooth model of $Sh_p(G, X)$ over $\mathcal{O}_L$; then S is a regular scheme.*

Proof. Let $s \in S$; we have to show that $R =_{df} \mathcal{O}_s$ is a regular local ring. Clearly $R = \varinjlim_K R_K$ where R_K is the local ring at the image of s in S_K, and the limit is taken over all K contained in K_0.

Set $R_0 = R_{K_0}$. Then clearly $R_0 \subset R \subset R_0^{\mathrm{sh}}$ where R_0^{sh} is a strict Henselization of R_0. To show that R is Noetherian, we use the following criterion: a ring R is Noetherian if every ascending chain

$$\mathfrak{a}_1 \subset \mathfrak{a}_2 \subset \mathfrak{a}_3 \subset \dots$$

of *finitely generated* ideals becomes constant. Since R_0^{sh} is Noetherian, the chain of ideals

$$\mathfrak{a}_1 R_0^{\mathrm{sh}} \subset \mathfrak{a}_2 R_0^{\mathrm{sh}} \subset \mathfrak{a}_3 R_0^{\mathrm{sh}} \subset \dots$$

becomes constant, and so it suffices to show that $\mathfrak{a} R_0^{\mathrm{sh}} \cap R = \mathfrak{a}$ for each finitely generated ideal $\mathfrak{a}$ of R. Clearly $\mathfrak{a} R_0^{\mathrm{sh}} \cap R \supset \mathfrak{a}$. Let $a_1, \dots, a_n$ generate $\mathfrak{a}$, and let $r \in \mathfrak{a} R_0^{\mathrm{sh}} \cap R$. After possibly replacing K_0 with a smaller compact open subgroup, we can assume that $a_1, \dots, a_n, r \in R_0$. Let $\mathfrak{a}_0$ be the ideal generated by $a_1, \dots, a_n$ in R_0. As R_0 is Noetherian, R_0^{sh} is faithfully flat over it, and so $\mathfrak{a}_0 R_0^{\mathrm{sh}} \cap R_0 = \mathfrak{a}_0$. But $\mathfrak{a}_0 R_0^{\mathrm{sh}} = \mathfrak{a} R_0^{\mathrm{sh}}$, and so $\mathfrak{a}_0 R_0^{\mathrm{sh}} \cap R_0$ contains r, which implies that $r \in \mathfrak{a}$.

Thus R is a Noetherian local ring, and R_0^{sh} is its strict Henselization. Therefore

$$\dim R = \dim R_0^{\text{sh}} = \dim R_0.$$

Because R_K is unramified over R_0 for all K, so also is R, and so any set of generators for the maximal ideal in R_0 also generates the maximal ideal in R. This proves that R is regular. □

Definition 2.5. A model S of $\text{Sh}_p(G,X)$ over $\mathcal{O}_L$ is said to have the ***extension property*** if, for every regular scheme Y over $\mathcal{O}_L$ such that Y_L is dense in Y, every L-morphism $Y_L \to S_L$ extends uniquely to an $\mathcal{O}_L$-morphism $Y \to S$.

Remark 2.6. Consider an inclusion $(G,X) \hookrightarrow (G',X')$ of pairs satisfying the axioms (SV1–4). Let K'_p be a compact open subgroup of $G'(\mathbb{Q}_p)$, and let $K_p = K'_p \cap G(\mathbb{Q}_p)$. Then $E(G,X) \supset E(G',X')$, and there is a closed immersion

$$\text{Sh}_p(G,X) \hookrightarrow \text{Sh}_p(G',X')$$

over $E(G,X)$, where $\text{Sh}_p(G,X)$ and $\text{Sh}_p(G',X')$ are the quotients of the canonical models of $\text{Sh}(G,X)$ and $\text{Sh}(G',X')$ by K_p and K'_p respectively. Let v be a prime of $E(G,X)$ lying over p. If $\text{Sh}_p(G',X')$ has a model S' over $\mathcal{O}_v$ with the extension property, then the closure of $\text{Sh}_p(G,X)$ in S' has the same property.

Langlands [Langlands 1976, p411], suggests that when K_p is hyperspecial and $\text{Sh}_p(G,X)_K$ is (proper and) smooth over E_v, then it should extend to a (proper and) smooth scheme over $\mathcal{O}_v$, but offers no suggestion of how to characterize the extended model. We suggest such a characterization.

Conjecture 2.7. *When K_p is hyperspecial, $\text{Sh}_p(G,X)$ has a smooth model over $\mathcal{O}_v$ with the extension property.*

Proposition 2.8. *There is at most one model of $\text{Sh}_p(G,X)$ over $\mathcal{O}_v$ satisfying the conditions of (2.7).*

Proof. Let S and S' be two such models. Because S' has the extension property and S is regular, the morphism

$$S_{E_v} \xrightarrow{\gamma} \text{Sh}_p(G,X)_{E_v} \xrightarrow{\gamma'^{-1}} S'_{E_v}$$

extends uniquely to a morphism $S \to S'$, and it is easy to see that this is an isomorphism compatible with the actions of $G(\mathbb{A}_f^p)$ and the maps γ. □

Definition 2.9. A smooth model $\mathrm{Sh}_p(G,X)_v$ of $\mathrm{Sh}_p(G,X)$ over $\mathrm{Spec}\,\mathcal{O}_v$ with the extension property will be called an ***integral canonical*** model.

Thus the integral canonical model is uniquely determined up to a unique isomorphism (if it exists).

Existence of canonical models: the Siegel modular variety. Let $\mathrm{Sh}(G,X)$ be the Siegel modular variety defined by a symplectic space (V,ψ) over $\mathbb{Q}$ (see 1.4). Let $V(\mathbb{Z}_p)$ be a $\mathbb{Z}_p$-lattice in $V(\mathbb{Q}_p)$ such that the discriminant of the restriction of ψ to $V(\mathbb{Z}_p)$ is a p-adic unit. Then

$$K_p =_{df} \{g \in G(\mathbb{Q}_p) \mid gV(\mathbb{Z}_p) = V(\mathbb{Z}_p)\}$$

is a hyperspecial subgroup of $G(\mathbb{Q}_p)$, and we set $\mathrm{Sh}_p(G,X) = \mathrm{Sh}(G,X)/K_p$.

Theorem 2.10. *The Siegel modular variety $Sh_p(G,X)$ has a canonical model over $\mathbb{Z}_p$.*

More precisely, we show that the moduli scheme over $\mathbb{Z}_p$ constructed by Mumford is a canonical model.

Proposition 2.11. *Let Y be a normal scheme, and let U be a dense open subscheme of Y. If A and B are abelian schemes on Y, then every homomorphism $\varphi\colon A|U \to B|U$ extends uniquely to a homomorphism $A \to B$.*

Proof. The uniqueness of the extension follows from fact that B is separated over Y. In proving the existence, we can assume Y to be affine and integral, say $Y = \mathrm{Spec}\,R$, and that φ is defined over some open subset $\mathrm{Spec}\,R_b$, $b \in R$. Every ring is a union of excellent rings, and the normalization of an excellent ring is Noetherian. Hence $R = \cup R_\alpha$ with R_α Noetherian and normal. For α sufficiently large, b will lie in R_α and φ will have a model φ_α over $\mathrm{Spec}(R_\alpha)_b$. Now we can apply [Faltings and Chai 1990, I.2.7], to obtain an extension of φ_α to $\mathrm{Spec}\,R_\alpha$, and this gives an extension of φ to $\mathrm{Spec}\,R$. □

Corollary 2.12. *Let $Y \supset U$ be as in the Proposition, and let A be an abelian scheme over U. If A extends to an abelian scheme on Y, then it does so uniquely.*

We next consider the existence of such an extension. Let Y be an integral scheme and let A be an abelian scheme over a dense open subscheme U of Y. Let η be the generic point of Y, and let $T_\ell(A_\eta)$ be the Tate module of A_η. There is an action of the étale fundamental group $\pi_1(U,\eta)$ on $T_\ell(A_\eta)$, and if A extends to an abelian scheme on Y, then this action factors through the quotient $\pi_1(Y,\eta)$ of $\pi_1(U,\eta)$. This statement has a converse.

Proposition 2.13. *Let Y be an integral regular scheme with generic point η. Assume that $\kappa(\eta)$ has characteristic zero, and let A be an abelian scheme over a dense open subscheme U of Y. If the monodromy representation $\pi_1(U,\eta) \to \mathrm{Aut}(T_\ell A_\eta)$ factors through $\pi_1(Y,\eta)$, then A extends to an abelian scheme on Y.*

Proof. Because of (2.12), there will be a largest open subscheme U of Y such that A extends to U. Suppose $U \neq Y$, and let $y \in Y$, $y \notin U$.

If $\mathcal{O}_y$ has dimension 1, then it is a discrete valuation ring. Its field of fractions is $\kappa(\eta)$, and the assumption implies that the action of $\mathrm{Gal}(\kappa(\eta)^{\mathrm{al}}/\kappa(\eta))$ on $T_\ell A$ factors through $\pi_1(\mathrm{Spec}\,\mathcal{O}_y,\eta)$. Now the Néron criterion of good reduction [Bosch et al. p183] implies that A_η extends to an abelian scheme on $\mathrm{Spec}\,\mathcal{O}_y$, which contradicts the maximality of U.

Therefore $Y-U$ has codimension ≥ 2 in Y, and [Faltings and Chai 1990, V.6.8] shows that A extends from U to the whole of Y. □

Let A be an abelian scheme over a scheme S, and let $A^\vee$ be the dual abelian scheme. An invertible sheaf $\mathcal{L}$ on A defines a morphism $\lambda(\mathcal{L})\colon A \to A^\vee$. A ***polarization*** of A is a homomorphism $\lambda\colon A \to A^\vee$ that is locally (for the étale topology on S) of the form $\lambda(\mathcal{L})$ for some ample invertible sheaf on A.

Proposition 2.14. *Let A be an abelian scheme on a normal scheme Y, and let $\lambda\colon A|U \to A^\vee|U$ be a polarization of $A|U$ for some dense open subset U of Y. Then λ extends to a polarization of A.*

Proof. We know (2.11) that λ extends to a homomorphism $\lambda\colon A \to A^\vee$, and the argument [Faltings and Chai 1990, p6: see 1.10b] shows that the extended λ will be a polarization. □

We now prove the theorem. Choose a $\mathbb{Z}$-lattice $V(\mathbb{Z})$ in V such that $V(\mathbb{Z})\otimes\mathbb{Z}_p = V(\mathbb{Z}_p)$, and for each N relatively prime to p, let

$$K(N) = \{g \in G(\mathbb{A}_f^p) \mid g \text{ acts as } 1 \text{ on } V(\mathbb{Z})/NV(\mathbb{Z})\}.$$

Then (over $\mathbb{C}$) $\mathrm{Sh}_p(G,X)/K(N)$ represents the functor that sends a $\mathbb{C}$-scheme Y to the set of isomorphism classes of triples (A,λ,η) where A is an abelian scheme over Y, λ is a polarization of A (taken up to a constant), and η is a level-N structure on A, i.e., an isomorphism $V(\mathbb{Z})/NV(\mathbb{Z}) \to A_N$. Mumford (1965) shows that the same functor is representable by a scheme $S(N)$ over $\mathbb{Z}_{(p)}$. The family $(S(N)_\mathbb{Q})$ is a canonical model for $\mathrm{Sh}_p(G,X)$ over $\mathbb{Q}$, and I claim that the family $(S(N))$ is an integral canonical model for $\mathrm{Sh}_p(G,X)$ over $\mathbb{Z}_p$. Since Mumford shows it to be smooth, it remains to prove that it has the extension property.

Consider a regular scheme Y over $\mathbb{Z}_p$ such that $U =_{df} Y_{\mathbb{Q}_p}$ is dense in Y. A morphism $\alpha\colon U \to \mathrm{Sh}_p(G,X)$ can be regarded as a projective system of morphisms $\alpha_N\colon U \to \mathrm{Sh}_p(G,X)/K(N)$, and the α_N's define a projective system of triples (A,λ,η_N) on U. Here (A,λ) is independent of N, and η_N is a level N-structure. The existence of level ℓ^m-structures for every m implies that $\pi_1(U,\eta)$ acts trivially on $T_\ell(A_\eta)$. Therefore (2.13) implies that A extends uniquely to an abelian scheme on Y, and (2.14) implies that λ extends uniquely to a polarization of over Y. Moreover, each level structure η_N extends. The universal property of $S(N)$ now implies that α_N extends uniquely to a morphism $Y \to S(N)$, and the inverse limit of these morphisms is an extension of α to Y.

Further remarks.

2.15. Let $\mathrm{Sh}(G,X)$ be a Shimura variety of Hodge type. After (2.6) and (2.10) we know that $\mathrm{Sh}_p(G,X)$ has an integral model with the extension property. To prove that the model is canonical, it remains to show that it is smooth. If $\mathrm{Sh}(G,X)$ is of PEL-type, this can (presumably) be done by identifying the model with a moduli scheme.

2.16. Consider the Shimura variety $\mathrm{Sh}(T,\{h\})$ defined by a torus. In order for $T(\mathbb{Q}_p)$ to have a hyperspecial subgroup K_p, T must split over $\mathbb{Q}_p^{\mathrm{un}}$, and then

$$K_p = (X_*(T) \otimes \mathcal{O}^\times)^{\mathrm{Gal}(\mathbb{Q}_p^{\mathrm{un}}/\mathbb{Q}_p)},$$

where $\mathcal{O}$ is the ring of integers in $\mathbb{Q}_p^{\mathrm{un}}$. It follows from the definitions that, for any compact open subgroup K^p of $G(\mathbb{A}_f^p)$, the canonical model of $\mathrm{Sh}(T,\{h\})/K^pK_p$ is of the form $\mathrm{Spec}\, R$ with R a product of fields L_i that are unramified over $E(T,\{h\})$ at v. The integral closure in R of the ring of integers in $E(T,\{h\})_v$ is an integral canonical model for $\mathrm{Sh}(T,\{h\})$.

2.17. For Shimura varieties of abelian type, it should be possible to prove the following result: choose an integral structure $G_{\mathbb{Z}}$ on G; for almost all p, $K_p =_{df} G_{\mathbb{Z}}(\mathbb{Z}_p)$ is a hyperspecial subgroup of $G(\mathbb{Q}_p)$, and for almost all of those p, $\mathrm{Sh}_p(G,X)$ has a canonical model over $\mathcal{O}_v$.

In fact, one really needs something stronger than (2.7).

Conjecture 2.18. *Assume K_p is hyperspecial; for every sufficiently small compact open subgroup K^p of $G(\mathbb{A}_f^p)$ there exists a smooth toroidal compactification of $\mathrm{Sh}_p(G,X)/K^p$ that extends to a smooth compactification of the integral canonical model.*

Notes. The material in this section is new.

3. THE PSEUDOMOTIVIC GROUPOID

The pseudomotivic groupoid is the groupoid (conjecturally) attached to the category of motives over $\mathbb{F}$. After discussing Weil numbers, we review the properties that the category of motives over $\mathbb{F}$ is expected to have, and we deduce information about the groupoid attached to the category. This motivates the definition of the pseudomotivic groupoid. Finally, we construct the map from the pseudomotivic groupoid to the Serre group that should correspond to the reduction of CM-motives to characteristic p.

Weil numbers. Let q be a power of p. A ***Weil q-integer*** is an algebraic integer π such that, for every embedding $\tau\colon \mathbb{Q}[\pi] \hookrightarrow \mathbb{C}$,

$$|\tau\pi| = q^{1/2}.$$

Two Weil numbers are said to be ***conjugate*** if they have the same minimum polynomial over $\mathbb{Q}$ or, equivalently, if there is an isomorphism of fields $\mathbb{Q}[\pi] \to \mathbb{Q}[\pi']$ carrying π to π'. The importance of Weil numbers is illustrated by the following two theorems. Recall that, for a variety X over $\mathbb{F}_q$, the ***Frobenius endomorphism*** π_X of X acts on $X(\mathbb{F})$ as

$$(x_0 : x_1 : \dots) \mapsto (x_0^q : x_1^q : \dots).$$

Theorem 3.1. *If A is a simple abelian variety over $\mathbb{F}_q$, then $\mathrm{End}_{\mathbb{F}_q}(A) \otimes \mathbb{Q}$ is a division algebra with centre $\mathbb{Q}[\pi_A]$. Moreover, π_A is a Weil q-integer, and the map $A \mapsto \pi_A$ determines a one-to-one correspondence between the set of isogeny classes of simple abelian varieties over $\mathbb{F}_q$ and the set of conjugacy classes of Weil q-integers.*

Proof. The statement combines theorems of Weil, Tate, and Honda. □

Theorem 3.2. *For any smooth complete variety X over $\mathbb{F}_q$, the characteristic polynomial of π_X acting on the étale cohomology group $H^i(X \otimes_{\mathbb{F}_q} \mathbb{F}, \mathbb{Q}_\ell)$ has coefficients in $\mathbb{Z}$, independent of $\ell \neq p$, and its roots are Weil q^i-integers.*

Proof. The statement combines theorems of Grothendieck and Deligne. □

Write $W(q)$ for the subgroup of $\mathbb{Q}^{\mathrm{al}\times}$ generated by the Weil q-integers. Note that every element of $W(q)$ is of the form π/q^m with π a Weil q-integer. If q' is a power of q, say $q' = q^m$, then $\pi \mapsto \pi^m$ is a homomorphism $W(q) \to W(q')$, and we set

$$W(p^\infty) = \varinjlim W(q).$$

An element of $W(q)$ will be referred to as a ***Weil number (for q).***

The category of motives over $\mathbb{F}$. Let k be an arbitrary field. For a smooth projective variety X of pure dimension d, let $Z^r(X)$ be the $\mathbb{Z}$-module of algebraic cycles on X of codimension r, and let $C^r(X)$ be the quotient of $Z^r(X) \otimes \mathbb{Q}$ by the subspace of cycles numerically equivalent to zero (i.e. such that $(Z \cdot Z') = 0$ for all $Z' \in Z^{d-r}(X)$). If Y is a second smooth projective variety over k, then the elements of $C^d(X \times Y)$ are called ***algebraic correspondences from X to Y of degree zero.*** For example, the graph of a morphism from Y to X defines an algebraic correspondence from X to Y of degree zero.

The ***category of correspondences*** $\mathbf{CV}^0(k)$ has one object $h(X)$ for each smooth projective variety X over k, and a morphism from $h(X)$ to $h(Y)$ is an algebraic correspondence of degree 0 from X to Y (see [Saavedra 1972, p384]). Then $X \mapsto h(X)$ is a contravariant functor, and $\mathbf{CV}^0(k)$ acquires the structure of a tensor category for which

$$h(X) \otimes h(Y) = h(X \times Y),$$

and the commutativity and associativity constraints are defined by the obvious isomorphisms

$$X \times Y \approx Y \times X, \quad X \times (Y \times Z) \approx (X \times Y) \times Z.$$

On adding the images of projectors and inverting the Lefschetz motive, one obtains the false category of motives $\mathbf{M}(k)$ over k (ibid. VI.4). This is a $\mathbb{Q}$-linear tensor category, but it can not be Tannakian because

$$\dim h(X) = (\Delta \cdot \Delta) = \sum (-1)^i \dim H^i_\ell(X),$$

which may be negative. In order to obtain a category that is (conjecturally) Tannakian, we must define a grading on it and use the grading to modify the commutativity constraint. For a general field, it is not proved that this is possible, but for a finite field or its algebraic closure, the following proposition allows us to do it.

Proposition 3.3. *Let X be a smooth projective variety over $\mathbb{F}_q$. There exist well-defined idempotents $p^i \in \mathrm{End}(h(X))$ such that $p^i H(X) = H^i(X)$ if $H(X)$ denotes ℓ-adic (étale) cohomology, $\ell \neq p$, or the crystalline cohomology.*

Proof. Let $P_i(T)$ be the characteristic polynomial of π_X acting on $H(X)$—it is known to lie in $\mathbb{Z}[T]$ and be independent of which cohomology theory we take.

According to the Cayley-Hamilton theorem, $P_i(\pi_X)$ acts as 0 on $H^i(X)$. Choose $P^i(T) \in \mathbb{Q}[T]$ such that

$$P^i(T) = \begin{cases} 1, & \text{mod } P_i(T); \\ 0, & \text{mod } P_j(T) \text{ for } j \neq i. \end{cases}$$

This is possible from the Chinese remainder theorem, because the P_i have roots of different absolute values and so must be relatively prime. Now take $p^i = P^i(\pi_X)$. □

The last result allows us to define a grading on $\mathbf{M}(\mathbb{F}_q)$ such that

$$h(X) = \oplus h(X)^i, \quad h(X)^i = \operatorname{Im}(p^i).$$

We can now modify the commutativity constraint in $\mathbf{M}(k)$ as follows: write the given commutativity constraint

$$\dot{\psi}_{M,N} \colon M \otimes N \to N \otimes M,$$

as a direct sum,

$$\dot{\psi}_{M,N} = \oplus \dot{\psi}^{r,s}, \quad \dot{\psi}^{r,s} \colon M^r \otimes N^s \xrightarrow{\approx} N^s \otimes M^r,$$

and define

$$\psi_{M,N} = \oplus (-1)^{rs} \dot{\psi}^{r,s}.$$

Now

$$\dim h(X) = \sum H^i_\ell(X). \tag{3.3.1}$$

In this way we obtain the ***category of motives*** $\mathbf{Mot}(\mathbb{F}_q)$ over $\mathbb{F}_q$.

Proposition 3.4. *The category* $\mathbf{Mot}(\mathbb{F}_q)$ *is a semisimple Tannakian category over* $\mathbb{Q}$.

Proof. Saavedra (1972), VI.4.1.3.5, shows that it is a $\mathbb{Q}$-linear tensor category with duals (i.e., satisfying A.7.2). It is obvious that $\operatorname{End}(1) = \mathbb{Q}$. Jannsen (1991) shows that $\mathbf{Mot}(\mathbb{F}_q)$ is a semisimple abelian category, and because the dimensions of its objects are nonnegative integers (see 3.3.1), Deligne (1990), 7.1, shows that it is Tannakian. □

Conjecture 3.5. *For each $\ell \neq p, \infty$, an algebraic cycle on X is numerically equivalent to zero if it maps to zero in the ℓ-adic étale cohomology of X; similarly, it is numerically equivalent to zero if it maps to zero in the crystalline cohomology of X.*

Remark 3.6. Conjecture (3.5) implies that there are fibre functors ω_ℓ, all $\ell \neq p, \infty$, and ω_{crys} such that

$$\omega_\ell(h(X)) = H^*_{\text{et}}(X \otimes \mathbb{F}, \mathbb{Q}_\ell), \quad \omega_{\text{crys}}(h(X)) = H^*_{\text{crys}}(X/W(\mathbb{F}_q)) \otimes B(\mathbb{F}_q),$$

for all smooth projective schemes X over $\mathbb{F}_q$.

We remark that similar constructions lead to a category $\mathbf{Mot}(\mathbb{F})$ of motives over $\mathbb{F}$; it is a Tannakian category over $\mathbb{Q}$, and when Conjecture 3.5 is assumed for projective smooth varieties over $\mathbb{F}$, then there are fibre functors ω_ℓ, all $\ell \neq p, \infty$, and ω_{crys}.

Polarizations. Consider a Tannakian category $\mathbf{T}$ over a subfield k of $\mathbb{R}$. For an object X of $\mathbf{T}$, a ***bilinear form*** on X is a mapping $\varphi\colon X \otimes X \to 1$. It is ***nondegenerate*** if the map $X \to X^\vee$ it defines is an isomorphism. The ***parity*** of φ is defined by the equation

$$\varphi(x, y) = \varphi(y, \varepsilon x).$$

Let $u \in \text{End}(X)$; the ***transpose*** u^t of u with respect to φ is defined by

$$\varphi(ux, y) = \varphi(x, u^t y).$$

The form φ is said to be a ***Weil form*** if ε is in the centre of $\text{End}(X)$ and if for all nonzero $u \in \text{End}(X)$, $\text{Tr}(u \cdot u^t) > 0$. Two Weil forms φ and ψ are said to be ***compatible*** if $\varphi \oplus \psi$ is also a Weil form.

Now let Z be the centre of the band attached to $\mathbf{T}$, and let $\varepsilon \in Z(\mathbb{R})$. Suppose there is given, for each X in $\mathbf{T}$, a compatibility class $\pi(X)$ of Weil forms on X with parity ε; we say that π is a ***polarization*** on $\mathbf{T}$ if, for all X and Y,

$$\varphi \in \pi(X), \quad \psi \in \pi(Y) \implies \varphi \oplus \psi \in \pi(X \oplus Y), \quad \varphi \otimes \psi \in \pi(X \otimes Y).$$

Example 3.7. Let $\mathbf{V}$ be the category of pairs (V, α) where V is a $\mathbb{Z}$-graded vector space over $\mathbb{C}$ and α is a semi-linear automorphism of V such that α^2 acts as $(-1)^n$ on the n^{th} graded piece of V. Then $\mathbf{V}$ has a natural tensor structure for which it is a nonneutral Tannakian category over $\mathbb{R}$. There is a canonical polarization π on $\mathbf{V}$: for V of even weight, $\pi(V, \alpha)$ consists of all symmetric positive definite forms on V; for V of odd weight, $\pi(V,\alpha)$ consists of all skew-symmetric positive definite forms on V (such a form is said to be positive definite if $\varphi(v,\alpha v)>0$ for all $v\neq 0$).

A ***Tate triple*** $(\mathbf{T},w,T)$ is Tannakian category together with a ***weight gradation*** $w\colon \mathbb{G}_m \to Aut^{\otimes}(\mathrm{id}_{\mathbf{T}})$ and a ***Tate object*** T of weight -2. There is a natural notion of a ***polarization*** of a Tate triple [Deligne and Milne 1982, p192]. For example, $\mathbf{V}$ with its natural gradation and polarization and the object $T=(\mathbb{C}, z\mapsto \bar{z})$ of weight -2 is a polarized Tate triple.

Remark 3.8. If $\mathbf{T}$ is polarizable, then $\mathrm{End}(X)$ is a semisimple k-algebra for all X in $\mathbf{T}$.

Proposition 3.9. *Let $(\mathbf{T},w,T)$ be a Tate triple over $\mathbb{R}$ such that $w(-1)\neq 1$, and let π be a polarization of $(\mathbf{T},w,T)$. Then there exists an exact faithful functor $\omega\colon \mathbf{T}\to\mathbf{V}$ preserving the Tate triple structures, and carrying π into the canonical polarization on $\mathbf{V}$; moreover, ω is uniquely determined up to isomorphism.*

Proof. See [Deligne and Milne 1982, 5.20]. (Unfortunately, the proof there is labyrinthine—a more direct proof would be useful.) □

Consequences of the standard conjectures. Fix a prime $\ell\neq p$. Let Z be a hyperplane section of X, and let z be the class of Z in $H^2_\ell(X)(1)$. Define

$$L\colon H^r_\ell(X)\to H^{r+2}_\ell(X)(1)$$

to be $a\mapsto a\cdot z$. The strong Lefschetz theorem (proved by Deligne in nonzero characteristic) states that for $r\leq d=\dim X$, the map

$$L^{d-r}\colon H^r_\ell(X)(r)\to H^{2d-r}_\ell(X)(2d-r)$$

is an isomorphism. Let $A^r(X)$ be the $\mathbb{Q}$-subspace of $H^{2r}_\ell(X)(r)$ generated by the algebraic cycles.

Conjecture 3.10. *For $2r\leq d=\dim X$, the injection*

$$L^{d-2r}\colon A^r(X)\to A^{d-r}(X)$$

is a bijection.

For $2r\leq d$, set

$$A^r_{\mathrm{prim}}(X)=\{\,a\in A^r(X)\mid L^{d-2r+1}a=0\}.$$

Conjecture 3.11. *For $2r \leq d = \dim X$, the quadratic form on $A^r_{\text{prim}}(X)$,*

$$a, b \mapsto (-1)^r < L^{d-2r} a \cdot b >$$

is positive definite.

These conjectures are due to Grothendieck.

Theorem 3.12. *Assume conjectures (3.5), (3.10), and (3.11). Then* $\mathbf{Mot}(\mathbb{F}_q)$ *has a canonical polarization.*

Proof. See [Saavedra 1972, VI.4.4]. □

When one assumes (3.5), (3.10), and (3.11), it is not necessary to use Jannsen's theorem, and by extension Deligne's theorem, to prove that $\mathbf{Mot}(\mathbb{F}_q)$ is a semisimple Tannakian category.

Again, a similar argument shows that $\mathbf{Mot}(\mathbb{F})$ is polarizable.

Definition 3.13. Let ω be a fibre functor for $\mathbf{Mot}(\mathbb{F}_q)$ over $\mathbb{Q}^{\text{al}}$, and let $\mathfrak{M}_q(\omega)$ be the corresponding groupoid. We call $\mathfrak{M}_q(\omega)$ the ***motivic groupoid*** attached to ω. When ω is understood, we drop it from the notations. Similarly, $\mathfrak{M}(\omega)$ is the ***motivic groupoid*** attached to a fibre functor ω on $\mathbf{Mot}(\mathbb{F})$.

Remark 3.14. The canonical functor $\mathbf{Mot}(\mathbb{F}_q) \to \mathbf{Mot}(\mathbb{F})$ is faithful and exact. Therefore a fibre functor on $\mathbf{Mot}(\mathbb{F})$ defines a fibre functor on $\mathbf{Mot}(\mathbb{F}_q)$ for every q. Correspondingly, we have morphisms of groupoids $\mathfrak{M} \to \mathfrak{M}_q$ for every q, and $\mathfrak{M} = \varprojlim \mathfrak{M}_q$.

Consequences of the Tate conjecture. In the present context, the Tate conjecture states the following.

Conjecture 3.15. *For any smooth projective variety X over $\mathbb{F}_q$, the $\mathbb{Q}_\ell$-subspace of $H^{2r}_\ell(X \otimes \mathbb{F}, \mathbb{Q}_\ell(r))$ generated by algebraic cycles is equal to $H^{2r}_\ell(X \otimes \mathbb{F}, \mathbb{Q}_\ell(r))^\Gamma$, where $\Gamma = \mathrm{Gal}(\mathbb{F}/\mathbb{F}_q)$.*

When combined with Conjecture 3.5, this implies that

$$\mathrm{Hom}(h(X), h(Y)) \otimes \mathbb{Q}_\ell \approx \mathrm{Hom}(H_\ell(X), H_\ell(Y))^\Gamma.$$

Theorem 3.16. *Assume the standard conjectures (3.10) and (3.11) and the Tate conjectures (3.5) and (3.15). If M is a simple motive over $\mathbb{F}_q$, then* $\mathrm{End}(M)$ *is a division algebra with centre $\mathbb{Q}[\pi_M]$. Moreover π_M is a Weil number for q, and the map $M \mapsto \pi_M$ determines a one-to-one correspondence between the set of* *isomorphism classes of simple motives varieties over $\mathbb{F}_q$ and the set of conjugacy classes of Weil numbers for q.*

Proof. If $\mathrm{End}(M)$ were not a division algebra, it would contain idempotents, and M would not be simple. Let F be the centre of $\mathrm{End}(M)$. Clearly $\mathbb{Q}[\pi_M] \subset F$. Conjecture (3.15) implies that these two algebras become equal when tensored with $\mathbb{Q}_\ell$, and this implies they are equal. The rest of the proof is straightforward. □

Corollary 3.17. *Under the hypotheses of theorem, $M_q =_{df} \mathfrak{M}_q^\Delta$ is the multiplicative group with character group $W(q)$, and $M =_{df} \mathfrak{M}^\Delta$ is the pro-torus with character group $W(p^\infty)$.*

Proof. The theorem implies that the simple objects of $\mathbf{Mot}(\mathbb{F}_q) \otimes \mathbb{Q}^{\mathrm{al}}$ are in one-to-one correspondence with the elements of $W(q)$, and hence that the affine group scheme attached to $\mathbf{Mot}(\mathbb{F}_q) \otimes \mathbb{Q}^{\mathrm{al}}$ and any fibre functor is the group of multiplicative type with character group $W(q)$. Now apply (A.13). □

From here through (3.26), we investigate the consequences of Conjectures (3.10), (3.11), (3.5), and (3.15).

Remark 3.18. There is a unique element $\delta_{nm} \in M_{p^n}(\mathbb{Q})$ such that $\chi_\pi(\delta_n) = \pi^m$ for all $\pi \in W(p^n)$, where χ_π is the character of M_{p^n} corresponding to π. Let M' be an algebraic quotient of M, and let M'_n be the quotient of M_n by the image of the kernel of $M \to M'$. Then $M \to M_n$ induces a homomorphism $M' \hookrightarrow M'_n$, and for n' sufficiently large, the image of $\delta_{n'}$ in $M'_n(\mathbb{Q})$ will lie in $M'(\mathbb{Q})$. We again denote in $\delta_{n'}$. If $n'' = mn'$, then $\delta_{n''} = \delta_{n'}^m$. The element $\delta_{n'}$ generates M' as an algebraic group over $\mathbb{Q}$. Note that, for any homomorphism $\varphi\colon M \to G$ from M into an algebraic group G, it makes sense to speak of $\varphi(\delta_n)$ for n sufficiently large.

Local study of $\mathfrak{M}$. We next want to study $\mathfrak{M}$ locally at each prime. For each ℓ, we choose a diagram

$$\begin{array}{ccc} \mathbb{Q}^{\mathrm{al}} & \longrightarrow & \mathbb{Q}_\ell^{\mathrm{al}} \\ \uparrow & & \uparrow \\ \mathbb{Q} & \longrightarrow & \mathbb{Q}_\ell, \end{array}$$

and we write $\mathfrak{M}(\ell)$ for the pull-back of $\mathfrak{M}$ relative to this diagram (see the discussion preceding A.5) and $\mathbf{Mot}(\mathbb{F}_q) \otimes \mathbb{Q}_\ell$ for the Tannakian category over $\mathbb{Q}_\ell$ obtained from $\mathbf{Mot}(\mathbb{F}_q)$ by extension of scalars (see A.12). (For $\ell = \infty$, $\mathbb{Q}_\ell = \mathbb{R}$.)

Study at ∞. The real Weil group $W(\mathbb{C}/\mathbb{R})$ is the extension

$$1 \to \mathbb{C}^\times \to W(\mathbb{C}/\mathbb{R}) \to \mathrm{Gal}(\mathbb{C}/\mathbb{R}) \to 1$$

defined by the cocycle

$$d_{1,1} = d_{1,\iota} = d_{\iota,1} = 1, \quad d_{\iota,\iota} = -1.$$

Define $\mathfrak{G}_\infty$ to be the $\mathbb{C}/\mathbb{R}$-groupoid with complex points $W(\mathbb{C}/\mathbb{R})$.

Proposition 3.19. *The groupoid attached to* $\mathbf{V}$ *and its canonical fibre functor is* $\mathfrak{G}(\infty)$.

Proof. Straightforward. □

Proposition 3.20. *There exists an exact faithful functor* $\omega_\infty\colon \mathbf{Mot}(\mathbb{F}_q) \otimes \mathbb{R} \to \mathbf{V}$ *preserving the Tate triple structure and carrying the canonical polarization of* $\mathbf{Mot}(\mathbb{F}_q) \otimes \mathbb{R}$ *into the canonical polarization on* $\mathbf{V}$*; moreover,* ω_∞ *is uniquely determined up to isomorphism.*

Proof. Combine (3.12) and (3.9). □

Corollary 3.21. *The homomorphism* $w\colon \mathbb{C}^\times \to M(\mathbb{C})$ *defined by the weight gradation on* $\mathbf{Mot}(\mathbb{F})$ *extends to a homomorphism of groupoids* $\zeta_\infty\colon \mathfrak{G}_\infty \to \mathfrak{M}(\infty)$*; the extension is unique up to isomorphism.*

Proof. Apply (3.19). □

Study at $\ell \neq p, \infty$. Let $\mathfrak{G}_\ell$ be the trivial $\mathbb{Q}_\ell^{\mathrm{al}}/\mathbb{Q}_\ell$-groupoid, i.e., the $\mathbb{Q}_\ell^{\mathrm{al}}/\mathbb{Q}_\ell$-groupoid such that

$$\mathfrak{G}_\ell(\mathbb{Q}_\ell^{\mathrm{al}}) = \mathrm{Gal}(\mathbb{Q}_\ell^{\mathrm{al}}/\mathbb{Q}_\ell).$$

Proposition 3.22. *There exists a homomorphism* $\zeta_\ell\colon \mathfrak{G}_\ell \to \mathfrak{M}(\ell)$*, well-defined up to isomorphism.*

Proof. As we noted above, the ℓ-adic étale cohomology defines a fibre functor ω_ℓ over $\mathbb{Q}_\ell$. The choice of an isomorphism $\omega \otimes_{\mathbb{Q}^{\mathrm{al}}} \mathbb{Q}_\ell^{\mathrm{al}} \to \omega_\ell \otimes_{\mathbb{Q}_\ell} \mathbb{Q}_\ell^{\mathrm{al}}$ defines a homomorphism ζ_ℓ. □

Study at p. Let k be a perfect field, let $W = W(k)$ be the ring of Witt vectors over k, and let $B = B(k)$ be the field of fractions of W. An ***isocrystal*** over k is a finite-dimensional vector space V over B together with a σ-linear isomorphism $\varphi\colon V \to V$. The category of isocrystals over k has a natural tensor structure for which it is a Tannakian category over $\mathbb{Q}_p$. The forgetful functor is a fibre functor over B.

Let L_n be the unramified extension of $\mathbb{Q}_p$ of degree n contained in $\mathbb{Q}_p^{\mathrm{al}}$, and let σ be the canonical generator of $\mathrm{Gal}(L_n/\mathbb{Q}_p)$. The fundamental class in

$H^2(\mathrm{Gal}(L_n/\mathbb{Q}_p), L_n^\times)$ is represented by a canonical cocycle $(d_{\rho,\tau})$ which takes the following values: for $0 \leq i, j < n$,

$$d_{\sigma^i,\sigma^j} = \begin{cases} p^{-1} & \text{if } i+j \geq n; \\ 1 & \text{otherwise.} \end{cases}$$

The Weil group $W(L_n/\mathbb{Q}_p)$ is the extension

$$1 \to L_n^\times \to W(L_n/\mathbb{Q}_p) \to \mathrm{Gal}(L_n/\mathbb{Q}_p) \to 1$$

corresponding to the above cocycle. Let $\mathfrak{D}'_n$ be the $L_n/\mathbb{Q}_p$-groupoid such that $\mathfrak{D}'_n(L_n) = W(L_n/\mathbb{Q}_p)$, and let $\mathfrak{D}_n$ be its pull-back to a $B/\mathbb{Q}_p$-groupoid. Thus $\mathfrak{D}_n$ has kernel $\mathbb{G}_m$ and it has a canonical section over $B \otimes_{\mathbb{Q}_p} B$. Whenever $m|n$ there is a homomorphism $\mathfrak{D}_n \to \mathfrak{D}_m$ (not preserving the canonical sections) whose restriction to the kernel is $a \mapsto a^{n/m}$, and on passing to the inverse limit we obtain an affine $B/\mathbb{Q}_p$-groupoid $\mathfrak{D}$ whose kernel is the pro-torus $\mathbb{G}$ with character group $\mathbb{Q}$.

Proposition 3.23. *The groupoid attached to the category of isocrystals over k and its forgetful fibre functor is $\mathfrak{D}$.*

Proof. Omitted. □

We call $\mathfrak{D}$ the ***Dieudonné groupoid***, and we write $\mathfrak{G}_p$ for its pull-back to a $\mathbb{Q}_p^{\mathrm{al}}/\mathbb{Q}_p$-groupoid. It is the groupoid attached to the fibre functor over $\mathbb{Q}_p^{\mathrm{al}}$, $V \mapsto V \otimes_B \mathbb{Q}_p^{\mathrm{al}}$. We also write $\mathfrak{G}_p^{(n)}$ for the pull-back of $\mathfrak{D}_n$ to a $\mathbb{Q}_p^{\mathrm{al}}/\mathbb{Q}_p$-groupoid.

Proposition 3.24. *The functor sending a smooth projective variety X to its crystalline cohomology $H_{\mathrm{crys}}(X)$ extends to a tensor functor from* $\mathbf{Mot}(\mathbb{F})$ *to the category of isocrystals over $\mathbb{F}$. Consequently, there is a homomorphism $\zeta_p\colon \mathfrak{G}_p \to \mathfrak{M}(p)$, well defined up to isomorphism.*

Proof. Straightforward. □

Summary. The following theorem summarizes the above discussion.

Theorem 3.25. *Assume the conjectures (3.5), (3.10), (3.11), and (3.15). Then the motives over $\mathbb{F}$ form a canonically polarized Tate triple. Choose a fibre functor ω and let $\mathfrak{M}(\omega)$ be the corresponding $\mathbb{Q}^{\mathrm{al}}/\mathbb{Q}$-groupoid. Then*

(a) *the kernel $M(\omega)$ of $\mathfrak{M}(\omega)$ is a pro-torus with character group $W(p^\infty)$;*
(b) *for each prime ℓ of $\mathbb{Q}$ (including p and ∞) there exist homomorphism $\zeta_\ell\colon \mathfrak{G}_\ell \to \mathfrak{M}(\omega)(\ell)$, well-defined up to isomorphism.*

If $\mathfrak{M}(\omega')$ is the groupoid corresponding to a second fibre functor ω' over $\mathbb{Q}^{\mathrm{al}}$, then the choice of an isomorphism $\omega \approx \omega'$ determines an isomorphism $\alpha\colon \mathfrak{M}(\omega) \to \mathfrak{M}(\omega')$ whose restriction to the kernel is the identity map, and α is well-defined up to isomorphism; for any ℓ, $\alpha(\ell)\circ\zeta_\ell \approx \zeta'_\ell$.

Remark 3.26. For each ℓ, the restriction of ζ_ℓ to the kernel has an explicit description, not involving motives, whose elaboration we leave to the reader.

Pseudomotivic groupoids. We now drop all assumptions. The above discussion suggests the following definition.

Definition 3.27. A ***pseudomotivic groupoid*** is a system $(\mathfrak{P}, (\zeta_\ell))$ consisting of a $\mathbb{Q}^{\mathrm{al}}/\mathbb{Q}$-groupoid $\mathfrak{P}$ and morphisms $\zeta_\ell\colon \mathfrak{G}_\ell \to \mathfrak{P}(\ell)$ for all ℓ satisfying the following conditions:

(a) the kernel P of $\mathfrak{P}$ is a pro-torus with character group $W(p^\infty)$;

(b) for each ℓ (including p and ∞), ζ_ℓ^Δ has the description hinted at in (3.26).

Theorem 3.28. *There exists a pseudomotivic groupoid $(\mathfrak{P}, (\zeta_\ell))$. If $(\mathfrak{P}', (\zeta'_\ell))$ is second pseudomotivic groupoid, then there is an isomorphism $\alpha\colon \mathfrak{P} \to \mathfrak{P}'$, such that $\zeta'_\ell \approx \alpha\circ\zeta_\ell$; moreover, α is uniquely determined up to isomorphism.*

Proof. One computes easily that the cohomology groups $H^1(\mathbb{Q}, P)$ and $\mathrm{Ker}^2(\mathbb{Q}, P)$ are both zero, and the theorem follows easily from this. □

Relation to CM-motives. Let **CM** be the category of motives of CM-type over $\mathbb{Q}^{\mathrm{al}}$ as defined, for example, in [Milne 1990, I.4]. It is a Tannakian category over $\mathbb{Q}$ with a canonical fibre functor H_B (Betti cohomology) over $\mathbb{Q}$. The associated group scheme is a pro-torus S called the ***Serre group***.

Conjecturally, a CM-motive M will have a model M_E over some number field E, and, after possibly replacing E with a larger field, M_E will have good reduction at the prime induced by the chosen embedding $\mathbb{Q}^{\mathrm{al}} \hookrightarrow \mathbb{Q}_p^{\mathrm{al}}$. After reducing modulo this prime, we will obtain a motive $M(p)$ over a subfield of $\mathbb{F}$. When regarded as motive over $\mathbb{F}$, $M(p)$ is independent of all choices. Therefore (conjecturally) we have a functor

$$M \mapsto M(p)\colon \mathbf{CM} \to \mathbf{Mot}(\mathbb{F}_q).$$

According to (A.10), this will give a homomorphism

$$\mathfrak{P} \to \mathfrak{G}_S,$$

well-defined up to isomorphism. In the remainder of this section, we construct such a homomorphism.

Remark 3.29. The Serre group is an inverse limit $S = \varprojlim S^L$ of tori S^L where L is a CM-field contained in $\mathbb{Q}^{\mathrm{al}}$ and S^L is a certain quotient of $(\mathbb{G}_m)_{L/\mathbb{Q}}$. For $n >> 1$, it is possible to define a Frobenius element $\gamma_n \in S^L(\mathbb{Q})$ as follows. The kernel of the canonical map $L^\times \to S^L(\mathbb{Q})$ contains a subgroup V of finite index in the group of units U of L. Let $\mathfrak{p}$ be the prime ideal of L induced by the embedding $L \subset \mathbb{Q}^{\mathrm{al}} \hookrightarrow \mathbb{Q}_p^{\mathrm{al}}$. For some m, $\mathfrak{p}^m$ is principal, say $\mathfrak{p}^m = (a)$. Let f be the degree of the residue extension $\kappa(\mathfrak{p})/\mathbb{F}_p$, and choose n so that n/mf is an integer killing U/V. Then the image γ_n of $a^{n/mf}$ in $S^L(\mathbb{Q})$ is independent of all choices.

Let T be a torus over a field k. When μ is a cocharacter of T defined over a finite Galois extension L of k, we set

$$\nu = \mathrm{Tr}_{L/k}\,\mu =_{df} \sum_{\tau \in \mathrm{Gal}(L/k)} \tau\mu.$$

It is a cocharacter of T rational over k.

Lemma 3.30. *Let $(d_{\rho,\tau})$ be a 2-cocycle for $\mathrm{Gal}(L/k)$ with values in $L^\times$, and let*

$$c_\rho = \prod_{t \in \mathrm{Gal}(L/k)} (\rho t \mu)(d_{\rho,t}).$$

Then (c_ρ) is a 1-cochain for $\mathrm{Gal}(L/k)$ with values in $T(L)$ having coboundary $(\nu(d_{\rho,\tau}))$, i.e.,

$$c_\rho \cdot \rho c_\tau \cdot c_{\rho\tau}^{-1} = \nu(d_{\rho,\tau}).$$

Proof. Direct calculation. □

Now assume k is a local field. We apply the above lemma to the Weil group of L/k:

$$1 \to L^\times \to W(L/k) \to \mathrm{Gal}(L/k) \to 1.$$

Choose a section $s\colon \mathrm{Gal}(L/k) \to W(L/k)$, and write $s(\rho)\cdot s(\tau) = d_{\rho,\tau} s(\rho\tau)$. Then $d_{\rho,\tau}$ is a 2-cocycle, and $\nu(d_{\rho,\tau})$ is split by the 1-cocycle (c_ρ) defined in the lemma. Consequently, there is a homomorphism of extensions:

$$W(L/k) \to T(L) \rtimes \mathrm{Gal}(L/k), \quad a \mapsto \nu(a), \quad a \in L^\times, \quad s(\rho) \mapsto (c_\rho, \rho).$$

Lemma 3.31. *Up to conjugation by an element of $T(L)$, the homomorphism just defined is independent of the choice of s.*

Proof. Direct calculation. □

Example 3.32.

(a) When we apply the above construction in the case that $k = \mathbb{R}$, $L = \mathbb{C}$ (taking s to be the canonical section), we see that any $\mu \in X_*(T)$ defines a homomorphism $\xi_\mu \colon W(\mathbb{C}/\mathbb{R}) \to T(\mathbb{C}) \rtimes \mathrm{Gal}(\mathbb{C}/\mathbb{R})$ such that

$$\xi_\mu(z) = \mu(z) \cdot (\iota\mu)(z), \quad z \in \mathbb{C}, \quad \xi_\mu(s(\iota)) = (\mu(-1), \iota).$$

We can regard ξ_μ as a homomorphism of groupoids $\mathfrak{G}_\infty \to \mathfrak{G}_T(\infty)$.

(b) When we apply the above construction in the case that $k = \mathbb{Q}_p$ and L_n is the unramified extension of $\mathbb{Q}_p$ of degree n contained in $\mathbb{Q}_p^{\mathrm{al}}$, we obtain from $\mu \in X_*(T)$ defined over L_n a homomorphism $W(L_n/\mathbb{Q}_p) \to T(L_n) \rtimes \mathrm{Gal}(L_n/\mathbb{Q}_p)$ whose restriction to the kernel is ν. These homomorphisms are compatible with varying n (up to isomorphism), and so define morphisms of groupoids

$$\xi_\mu \colon \mathfrak{D} \to \mathfrak{G}_T(p), \quad \xi_\mu \colon \mathfrak{G}_p \to \mathfrak{G}_T(p).$$

Theorem 3.33. *For any algebraic quotient S' of the Serre group, there exists a morphism $\varphi \colon \mathfrak{P} \to \mathfrak{G}_{S'}$ such that*

(i) $\varphi(\delta_n) = \gamma_n\ , \quad n >> 1;$

(ii) $\varphi(\ell) \circ \zeta_\ell \approx \begin{cases} \xi_\mu \ \textit{if}\ \ell = \infty, \\ \xi_{-\mu} \ \textit{if}\ \ell = p, \\ \xi \ \textit{otherwise, where}\ \xi\ \textit{is the canonical splitting}\ \mathfrak{G}_\ell \to \mathfrak{G}_{S'}. \end{cases}$

Moreover, φ is uniquely determined by these conditions up to isomorphism.

Proof. Omitted. □

Corollary 3.34. *Let T be a torus over $\mathbb{Q}$ split by a CM-field, and let μ be a cocharacter of T whose weight is defined over $\mathbb{Q}$. Then there is a homomorphism*

$$\varphi_\mu \colon \mathfrak{P} \to \mathfrak{G}_T,$$

well defined up to isomorphism.

Proof. Under the hypotheses on (T, μ), there is a unique homomorphism $\rho \colon S \to T$ carrying the canonical cocharacter of S to μ. Clearly ρ will factor through some algebraic quotient S' of S, and so it defines a homomorphism of groupoids $\mathfrak{G}_{S'} \to \mathfrak{G}_T$. We define φ_μ to be the composite this with φ. □

Unramified homomorphisms. For use in the next section, we state some results concerning homomorphisms $\mathfrak{G}_p \to \mathfrak{G}_G$. Recall that we have an exact sequence

$$1 \to \mathbb{Q}_p^{\mathrm{al}\times} \to \mathfrak{G}_p^{(n)}(\mathbb{Q}_p^{\mathrm{al}}) \to \mathrm{Gal}(\mathbb{Q}_p^{\mathrm{al}}/\mathbb{Q}_p) \to 1,$$

and a canonical section s. A homomorphism $\theta\colon \mathfrak{G}_p^{(n)} \to \mathfrak{G}_G$ defines a homomorphism

$$\mathfrak{G}_p^{(n)}(\mathbb{Q}_p^{\mathrm{al}}) \to G(\mathbb{Q}_p^{\mathrm{al}}) \rtimes \mathrm{Gal}(\mathbb{Q}_p^{\mathrm{al}}/\mathbb{Q}_p),$$

and we say that θ is ***unramified*** if

$$\rho|\mathbb{Q}_p^{\mathrm{un}} = \mathrm{id} \implies \theta(s(\rho)) = (1, \rho).$$

Proposition 3.35.

(a) *An unramified homomorphism arises by pull-back from a (unique) homomorphism $\theta'\colon \mathfrak{D}_n \to \mathfrak{G}_G$.*

(b) *Every homomorphism $\theta\colon \mathfrak{G}_p^{(n)} \to \mathfrak{G}_G$ is isomorphic to an unramified homomorphism θ', and θ' is uniquely determined up to conjugation by an element of $G(\mathbb{Q}_p^{\mathrm{un}})$.*

Proof. Omitted. □

Remark 3.36. Consider a homomorphism $\theta\colon \mathfrak{G}_p \to \mathfrak{G}_G$. For some n, θ will factor into

$$\mathfrak{G}_p \to \mathfrak{G}_p^{(n)} \xrightarrow{\theta^{(n)}} \mathfrak{G}_G,$$

and $\theta^{(n)}$ will be equivalent to an unramified homomorphism $\theta'\colon \mathfrak{D}_n \to \mathfrak{G}_G$. Set $\theta'(s(\sigma)) = (b(\theta), \sigma)$. From (a) of the proposition, we know that $b(\theta) \in G(B)$, and from (b) of the proposition, we know that it is uniquely determined by θ up to σ-conjugacy. Thus we have a well-defined map

$$\theta \mapsto [b(\theta)]\colon \mathrm{Hom}(\mathfrak{G}_p, \mathfrak{G}_G) \to \mathbb{B}(G)$$

where $\mathbb{B}(G)$ is the set of σ-conjugacy classes of elements in $G(B)$ (see Appendix B).

Notes. Theorems 3.25, 3.28, and 3.33 are proved in [Langlands and Rapoport 1987]. The article [Milne 1991b] is an expanded version of this section.

4. STATEMENT OF THE MAIN CONJECTURE

Throughout this section, $\mathrm{Sh}(G,X)$ is a Shimura variety whose weight is defined over $\mathbb{Q}$. Let $E = E(G,X)$, and let v be a prime of E lying over the rational prime p. Since we are only interested in the case of good reduction, we assume that there is a hyperspecial group K_p in $G(\mathbb{Q}_p)$, and we set $\mathrm{Sh}_p(G,X) = \mathrm{Sh}(G,X)/K_p$ where $\mathrm{Sh}(G,X)$ is the canonical model over $E(G,X)$. We assume that $\mathrm{Sh}_p(G,X)$ has an integral canonical model $\mathrm{Sh}_p(G,X)_v$ over $\mathcal{O}_v$ (see 2.9).

Throughout, we fix an algebraic closure $\mathbb{Q}_p^{\mathrm{al}}$ of E_v, and we extend the inclusion of E into E_v to an inclusion of $\mathbb{Q}^{\mathrm{al}}$ into $\mathbb{Q}_p^{\mathrm{al}}$. For each $\ell \neq p, \infty$, we choose an algebraic closure $\mathbb{Q}_\ell^{\mathrm{al}}$ of $\mathbb{Q}_\ell$ and an embedding of $\mathbb{Q}^{\mathrm{al}}$ into $\mathbb{Q}_\ell^{\mathrm{al}}$. Finally we choose a pseudomotivic groupoid $(\mathfrak{P}, (\zeta_\ell))$, as in (3.27).

The set $\mathrm{Sh}_p(\mathbb{F}) =_{df} \mathrm{Sh}_p(G,X)_v(\mathbb{F})$ has an action of $G(\mathbb{A}_f^p)$ and an action of the geometric Frobenius element Φ of $\mathrm{Gal}(\mathbb{F}/\kappa(v))$. Because the action of $G(\mathbb{A}_f^p)$ is defined over $\mathcal{O}_v$, these two actions commute. The conjecture describes the isomorphism class of the set $\mathrm{Sh}_p(\mathbb{F})$ together with these commuting actions, which we abbreviate to $(\mathrm{Sh}_p(\mathbb{F}), \Phi, \times)$.

The set with operators $S(\varphi)$. Let φ be a homomorphism $\mathfrak{P} \to \mathfrak{G}_G$. We explain how to attach to φ a set $S(\varphi)$ together commuting actions of a Frobenius element $\Phi(\varphi)$ and $G(\mathbb{A}_f^p)$. For each ℓ (including p and ∞) we obtain by pull-back a morphism $\varphi(\ell)\colon \mathfrak{P}(\ell) \to \mathfrak{G}_G(\ell)$, and we write θ_ℓ for the composite of this with $\zeta_\ell\colon \mathfrak{G}_\ell \to \mathfrak{P}(\ell)$; thus

$$\theta_\ell\colon \mathfrak{G}_\ell \xrightarrow{\varphi(\ell)\circ\zeta_\ell} \mathfrak{G}_G(\ell).$$

Let $I_\varphi = Aut(\varphi)$. It is an algebraic group over $\mathbb{Q}$ such that

$$I_\varphi(\mathbb{Q}) = \{g \in G(\mathbb{Q}^{\mathrm{al}}) \mid \mathrm{ad}\, g \circ \varphi = \varphi\}.$$

Moreover,

$$(I_\varphi)(\mathbb{Q}_\ell) = \mathrm{Aut}(\varphi(\ell)) \subset \mathrm{Aut}(\theta_\ell).$$

Consider a prime $\ell \neq p, \infty$. There is a canonical homomorphism $\xi_\ell\colon \mathfrak{G}_\ell \to \mathfrak{G}_G$, which on points is the obvious section to

$$G(\mathbb{Q}_\ell^{\mathrm{al}}) \rtimes \mathrm{Gal}(\mathbb{Q}_\ell^{\mathrm{al}}/\mathbb{Q}_\ell) \to \mathrm{Gal}(\mathbb{Q}_\ell^{\mathrm{al}}/\mathbb{Q}_\ell),$$

and (see A.17)

$$G_{\mathbb{Q}_\ell} = Aut(\xi_\ell).$$

Define

$$X_\ell(\varphi) = \mathrm{Isom}(\xi_\ell, \theta_\ell) =_{df} \{g \in G(\mathbb{Q}_\ell^{\mathrm{al}}) \mid \mathrm{ad}\, g \circ \xi_\ell = \theta_\ell\}.$$

Then $I_\varphi(\mathbb{Q})$ acts on $X_\ell(\varphi)$ on the left, and $G(\mathbb{Q}_\ell)$ acts on $X_\ell(\varphi)$ on the right. If $X_\ell(\varphi)$ is nonempty, this second action makes $X_\ell(\varphi)$ into a principal homogeneous space.

Choose a $\mathbb{Z}$-structure on G, and let $X'_\ell(\varphi)$ be the subset of $X_\ell(\varphi)$ of integral elements. Define $X^p(\varphi)$ to be the restricted product of the $X_\ell(\varphi)$, $\ell \neq p, \infty$, relative to the subsets $X'_\ell(\varphi)$. It is independent of the choice of the $\mathbb{Z}$-structure. The group $G(\mathbb{A}_f^p)$ acts on $X^p(\varphi)$ on the right (and makes it into a principal homogeneous space if nonempty), and $I_\varphi(\mathbb{Q})$ acts on it on the left.

Consider $\theta_p \colon \mathfrak{G}_p \to \mathfrak{G}_G(p)$. As is explained in (3.36), this will factor through $\mathfrak{G}_p^{(n)}$ for some n,

$$\mathfrak{G}_p \longrightarrow \mathfrak{G}_p^{(n)} \xrightarrow{\theta_p^{(n)}} \mathfrak{G}_G(p),$$

and $\theta_p^{(n)}$ will be isomorphic to an unramified homomorphism $\theta' \colon \mathfrak{G}_p^{(n)} \to \mathfrak{G}_G(p)$. Set $\theta'(s(\sigma)) = (b(\theta_p), \sigma)$. Then $b(\theta_p) \in G(B)$, and it is well-defined up to σ-conjugacy by an element of $G(B)$.

From the definition of $E(G,X)$ we know that X defines a $G(\mathbb{Q}^{\mathrm{al}})$-conjugacy class of cocharacters $\mathfrak{c}(X)_{\mathbb{Q}^{\mathrm{al}}}$ of $G_{\mathbb{Q}^{\mathrm{al}}}$ that is stable under the action of $\mathrm{Gal}(\mathbb{Q}^{\mathrm{al}}/E)$ (see the discussion following 1.7). Using the embedding $\mathbb{Q}^{\mathrm{al}} \hookrightarrow \mathbb{Q}_p^{\mathrm{al}}$ we transfer $\mathfrak{c}(X)_{\mathbb{Q}^{\mathrm{al}}}$ to a conjugacy class $\mathfrak{c}(X)_{\mathbb{Q}_p^{\mathrm{al}}}$ of cocharacters of $G_{\mathbb{Q}_p^{\mathrm{al}}}$ stable under $\mathrm{Gal}(\mathbb{Q}_p^{\mathrm{al}}/E_v)$. Because G splits over B, $\mathfrak{c}(X)_{\mathbb{Q}^{\mathrm{al}}}$ arises from a $G(B)$-conjugacy class $\mathfrak{c}(X)_B$ of cocharacters of G_B. The group $G(B) \rtimes \mathrm{Gal}(B/\mathbb{Q}_p)$ acts on the building, and we set

$$X_p(\varphi) = \{x \in \mathcal{V}(B) \mid \mathrm{inv}(Fx, x) = \mathfrak{c}(X)_B\}$$

where $F = \theta'(s(\sigma)) = (b(\theta_p), \sigma)$.

Alternatively, let $C_p \in G(W)\backslash G(B)/G(W)$ be the double coset corresponding to $\mathfrak{c}(X)_B$ (see the subsection on hyperspecial groups at the end of §1); thus

$$C_p = G(W) \cdot \mu(p) \cdot G(W)$$

where μ is a cocharacter representing $\mathfrak{c}(X)_B$ and factoring through a torus S corresponding to K_p. Then

$$X_p(\varphi) = \{g \in G(B)/G(W) \mid g^{-1} \cdot b \cdot \sigma g \in C_p\}$$

where $b = b(\theta_p)$.

There is a natural action of $\mathrm{Aut}(\theta')$ on X_p on the left, and the choice of an isomorphism $\theta_p^{(n)} \to \theta'$ allows us to transfer this to an action of $I_\varphi(\mathbb{Q})$.

For $g \in X_p(\varphi)$, define

$$\Phi g = F^m g =_{df} b\sigma(b\sigma(\ldots g)) = b \cdot \sigma b \cdot \sigma^2 b \cdot \ldots \cdot \sigma^m g,$$

where $m = [E_v\colon \mathbb{Q}_p]$. Then

$$(\Phi g)^{-1} \cdot b \cdot \sigma(\Phi g) = \sigma^m(g^{-1} \cdot b \cdot \sigma g) \in \sigma^m C_p = C_p$$

because $\mathfrak{c}(X)_B$ (and hence C_p) is stable under $\mathrm{Gal}(B/E_v)$. Note that we can also write $\Phi g = \mathcal{N} b \cdot \sigma^m g$ where $\mathcal{N} b = b \cdot \sigma b \cdot \ldots \cdot \sigma^{m-1} b$.

Define

$$S(\varphi) = \varprojlim I_\varphi(\mathbb{Q}) \backslash (X^p(\varphi)/K^p) \times X_p(\varphi),$$

where the limit is over the compact open subgroups K^p of $G(\mathbb{A}_f^p)$. The group $G(\mathbb{A}_f^p)$ acts on $S(\varphi)$ through its action on $X^p(\varphi)$, and $\Phi(\varphi)$ acts through the action of Φ on $X_p(\varphi)$.

Lemma 4.1. *The isomorphism class of the triple $(S(\varphi), \Phi(\varphi), \times(\varphi))$ depends only on the isomorphism class of φ.*

Proof. Let $\varphi' \colon \mathfrak{P} \to \mathfrak{G}_G$ be a second homomorphism, and assume that $\varphi' = \mathrm{ad}\, c \circ \varphi$ for some $c \in G(\mathbb{Q}^{\mathrm{al}})$. For some n, $\theta_p =_{df} \varphi(p) \circ \zeta_p$ and $\theta'_p =_{df} \varphi'(p) \circ \zeta_p$ will factor through $\mathfrak{G}_p^{(n)}$. Choose an h and h' in $G(\mathbb{Q}_p^{\mathrm{al}})$ such that $\theta =_{df} \mathrm{ad}\, h \circ \theta_p^{(n)}$ and $\theta' = \mathrm{ad}\, h' \circ \theta_p'^{(n)}$ are unramified. Then $\theta' = \mathrm{ad}(h'') \circ \theta$ where $h'' = h' \cdot c \cdot h^{-1}$. It follows easily from the fact that θ and θ' are unramified that $h'' \in G(B)$.

We have bijections:

$$\begin{aligned} I_\varphi(\mathbb{Q}) &\to I_{\varphi'}(\mathbb{Q}), \quad g \mapsto \mathrm{ad}(c)(g); \\ X_\ell(\varphi) &\to X_\ell(\varphi'), \quad x^p \mapsto cx^p, \quad \ell \neq p, \infty; \\ X_p(\varphi) &\to X_p(\varphi'), \quad x_p \mapsto h'' x_p. \end{aligned}$$

On combining the last two bijections, we obtain a bijection

$$X^p(\varphi) \times X_p(\varphi) \to X^p(\varphi') \times X_p(\varphi')$$

which is equivariant relative to the isomorphism $I_\varphi(\mathbb{Q}) \to I_{\varphi'}(\mathbb{Q})$ (left actions) and for the right actions of $G(\mathbb{A}_f^p)$. We therefore obtain a $G(\mathbb{A}_f^p)$-equivariant bijection

$$[x^p, x_p] \mapsto [cx^p, h'' x_p] \colon S(\varphi) \to S(\varphi').$$

Write $\theta(s(\sigma)) = (b,\sigma)$ and $\theta'(s(\sigma)) = (b',\sigma)$. Then $b' = h'' \cdot b \cdot \sigma(h'')^{-1}$, and so $\mathcal{N}b' = h'' \cdot \mathcal{N}b \cdot \sigma^m(h'')^{-1}$. By definition, $\Phi[x^p, x_p] = [x^p, \mathcal{N}b \cdot \sigma^m x_p]$, which maps to

$$[cx^p, h'' \cdot \mathcal{N}b \cdot \sigma^m x_p] = [cx^p, \mathcal{N}b' \cdot \sigma^m(h'' x_p)] = \Phi'[cx^p, h'' x_p].$$

Thus the bijection commutes with the actions of the Frobenius elements. □

The conjecture will take the form that $(\mathrm{Sh}_p(\mathbb{F}), \Phi, \times)$ is isomorphic to a disjoint union of sets with operators of the form $(S(\varphi), \Phi(\varphi), \times(\varphi))$. The difficulty is in determining the indexing set for the union. It will be illuminating to look first at the case of the Shimura variety defined by a torus.

The case of a torus. Let T be a torus over $\mathbb{Q}$ that splits over a CM-field, and let $h\colon \mathbb{S} \to T$ be a homomorphism whose weight is defined over $\mathbb{Q}$. Let μ be the cocharacter of T associated with h, and let $E = E(T, \{h\})$ be the field of definition of μ; it is the reflex field of $\mathrm{Sh}(T, \{h\})$. Choose a prime number p and a prime v of E that is unramified over p. Assume that T splits over an unramified extension of $\mathbb{Q}_p$, and let K_p be the hyperspecial subgroup of $T(\mathbb{Q}_p)$ defined in (2.16). Then, as we noted in (2.16), $\mathrm{Sh}_p(T, \{h\})$ has a canonical model over $\mathcal{O}_v$.

Proposition 4.2. *Let $\varphi = \varphi_\mu\colon \mathfrak{P} \to \mathfrak{G}_T$ be the homomorphism defined in (3.34). Then the sets with operators $(Sh_p(\mathbb{F}), \Phi, \times)$ and $(S(\varphi), \Phi(\varphi), \times(\varphi))$ are isomorphic.*

Proof. The maps

$$\mathrm{Sh}_p(B) \leftarrow \mathrm{Sh}_p(\mathcal{O}_v) \to \mathrm{Sh}_p(\mathbb{F})$$

are bijective because $\mathrm{Sh}_p(T, \{h\})$ is of dimension zero and pro-étale over $\mathcal{O}_v$. Thus

$$\mathrm{Sh}_p(\mathbb{F}) = T(\mathbb{Q})^- \backslash T(\mathbb{A}_f^p) \times (T(\mathbb{Q}_p)/T(\mathbb{Z}_p)), \quad T(\mathbb{Z}_p) = K_p.$$

The cocharacter μ is defined over E, and defines a homomorphism

$$r\colon (\mathbb{G}_m)_{E/\mathbb{Q}} \xrightarrow{(\mu)_{E/\mathbb{Q}}} \mathrm{Res}_{E/\mathbb{Q}}(T_E) \xrightarrow{\mathrm{Nm}} T.$$

Let

$$\pi_v = (1, \dots, 1, p, 1, \dots, 1), \quad (p \text{ in } v\text{th position}).$$

Then, according to Deligne's convention, $\mathrm{rec}_E(s)$ acts on E^{ab} as the geometric Frobenius element at v. Therefore Φ acts on $\mathrm{Sh}_p(\mathbb{F})$ as multiplication by

$$r(\pi_v) = (1, \dots, 1, \mathrm{Nm}_{E_v/\mathbb{Q}_p} \mu(p), 1, \dots, 1), \quad \mathrm{Nm}\, p \text{ in the } p\text{th position}).$$

This gives a complete description of $(\mathrm{Sh}_p(\mathbb{F}), \Phi, \times)$. To obtain a similar description of $(S(\varphi), \Phi(\varphi), \times)$ we shall need the following lemma.

Lemma 4.3. *Let T be an unramified torus over $\mathbb{Q}_p$. For any $\mu \in X_*(T)$, we have a homomorphism $\xi_\mu\colon W(B/\mathbb{Q}_p) \to T(B) \rtimes \mathrm{Gal}(B/\mathbb{Q}_p)$ well-defined up to conjugation by an element of $T(B)$ (see 3.32b). Write $\xi_\mu(s(\sigma)) = (b_\mu, \sigma)$. Then $[b_\mu] = [\mu(p^{-1})]$ in $\mathbb{B}(T)$.*

Proof. Take n large enough so that T is split by the extension L_n of degree n of $\mathbb{Q}_p$ contained in B. Let $(d_{\rho,\tau})$ be the canonical fundamental cocycle for $\mathrm{Gal}(L_n/\mathbb{Q}_p)$ with values in $L_n^\times$ (see the discussion preceding 3.23). For any $\rho \in \mathrm{Gal}(\mathbb{Q}_p^{\mathrm{al}}/\mathbb{Q}_p)$, set

$$c_\rho = \prod_{t \in \mathrm{Gal}(L_n/\mathbb{Q}_p)} (\rho t \mu)(d_{\rho,t})$$

where, in the product, ρ denotes $\rho|L_n$. Then

$$a \cdot s(\rho) \mapsto \nu(a)(c_\rho, \rho)\colon \mathfrak{D}_n \to \mathfrak{G}_T,$$

is an unramified homomorphism whose composite with $\mathfrak{D} \to \mathfrak{D}_n$ represents ξ_μ. Therefore

$$b_\mu = c_\sigma =_{df} \prod_{0 \le i \le n-1} (\sigma^{i+1}\mu)(d_{\sigma,\sigma^i}).$$

But all the terms in this product are 1 except for that corresponding to $i = n-1$, which has the value $\mu(p^{-1})$, as required. $\square$

The group $I_\varphi = T$. The choice of an isomorphism $\alpha_0\colon \xi_\ell \to \theta_\ell$ defines a bijection

$$T(\mathbb{Q}_\ell) \to X_\ell(\varphi), \quad g \mapsto \alpha_0 \circ g.$$

Moreover,

$$X_p(\varphi) = \{g \in T(B)/T(W) \mid g^{-1} \cdot b \cdot \sigma g \in \mu(p) \cdot T(W)\}.$$

The above lemma shows that we can take b to be $\mu(p)$ (recall that $\varphi(p) \circ \zeta_p \approx \xi_{-\mu}$), and so

$$X_p(\varphi) = (T(B)/T(W))^\Gamma$$

where $\Gamma = \mathrm{Gal}(B/\mathbb{Q}_p)$. Now the cohomology sequence

$$0 \to T(\mathbb{Z}_p) \to T(\mathbb{Q}_p) \to (T(B)/T(W))^\Gamma \to H^1(\Gamma, T(W)) = 0$$

shows that $X_p(\varphi) = T(\mathbb{Q}_p)/T(\mathbb{Z}_p)$.

It remains to compute the action of $\Phi(\varphi)$, but this is multiplication by $\mathrm{Nm}_{E_v/\mathbb{Q}_p}\, \mu(p)$, and so agrees with the action of $r(\pi_v)$. $\square$

Statement of the main conjecture. A point $x \in X$ is said to be ***special*** if h_x factors through $T_{\mathbb{R}}$ for some torus $T \subset G$. For each such pair (T, x), we obtain a homomorphism (see 3.34)

$$\varphi_x = \varphi_{\mu_x} : \mathfrak{P} \longrightarrow \mathfrak{G}_T \subset \mathfrak{G}_G$$

which, as the notation suggests, is independent of the choice of T. Call such a homomorphism ***special***.

Main Conjecture 4.4. *There is an isomorphism*

$$(Sh_p(\mathbb{F}), \Phi, \times) \approx \coprod_{\varphi} (S(\varphi), \Phi(\varphi), \times(\varphi)) \tag{4.4.1}$$

where the disjoint union is taken over a set of representatives for the isomorphism classes of special homomorphisms $\varphi \colon \mathfrak{P} \to \mathfrak{G}_G$.

We call a homomorphism $\varphi \colon \mathfrak{P} \to \mathfrak{G}_G$ ***admissible*** if it is isomorphic to a homomorphism of the form φ_x, x special, and the set $S(\varphi)$ is nonempty. Then we could restate the conjecture as saying that there is a bijection (4.4.1) with the φ's running over a set of representatives for the isomorphism classes of admissible homomorphisms.

Admissible homomorphisms. We give a criterion for admissibility which applies in the case that G^{der} is simply connected.

Recall that $\mathfrak{G}_\infty$ is the $\mathbb{C}/\mathbb{R}$ groupoid such that $\mathfrak{G}_\infty(\mathbb{C}) = W(\mathbb{C}/\mathbb{R})$ (the real Weil group). Thus there is an exact sequence

$$1 \to \mathbb{C}^\times \to \mathfrak{G}_\infty(\mathbb{C}) \to \mathrm{Gal}(\mathbb{C}/\mathbb{R}) \to 1$$

and a canonical section $s\colon \mathrm{Gal}(\mathbb{C}/\mathbb{R}) \to \mathfrak{G}_\infty(\mathbb{C})$. Recall also that the neutral gerb $\mathfrak{G}_G$ has points $\mathfrak{G}_G(\mathbb{C}) = G(\mathbb{C}) \rtimes \mathrm{Gal}(\mathbb{C}/\mathbb{R})$.

Lemma 4.5. *For any point $x \in X$, the formulas*

$$\xi_x(z) = (w_X(z), \mathrm{id}), \quad \xi_x(s(\iota)) = (\mu_x(-1)^{-1}, \iota)$$

define a morphism of $\mathbb{C}/\mathbb{R}$-groupoids. If $x' = gx$, $g \in G(\mathbb{R})$, then $\xi_{x'} = \mathrm{ad}\, g \circ \xi_x$.

Proof. To show that the formulas define a homomorphism of abstract groups, it suffices to verify that $\xi_x(s(\iota))^2 = \xi_x(-1)$. But

$$(\mu_x(-1)^{-1}, \iota)^2 = ((\mu_x(-1) \cdot \iota(\mu_x(-1)^{-1}), \mathrm{id}) = (w_X(-1), \mathrm{id})$$

as required. It is now obvious that the formulas define a morphism of groupoids. The second statement is obvious. □

We write ξ_∞ for ξ_x, any x.

When φ is a homomorphism $\mathfrak{P} \to \mathfrak{G}_G$, we write φ^{ab} for the composite of φ with the map $\mathfrak{G}_G \to \mathfrak{G}_{G^{\mathrm{ab}}}$ induced by the quotient map $G \to G^{\mathrm{ab}}$.

Theorem 4.6. *Assume that G^{der} is simply connected. A homomorphism $\varphi\colon \mathfrak{P} \to \mathfrak{G}_G$ is admissible if and only if it satisfies the following conditions:*

(a) *$\zeta_\infty \circ \varphi(\infty)$ is isomorphic to ξ_∞;*

(b) *$\zeta_\ell \circ \varphi(\ell)$ is isomorphic to ξ_ℓ;*

(c) *the set $X_p(\varphi)$ is nonempty;*

(d) *the composite φ^{ab} of φ with $\mathfrak{G}_G \to \mathfrak{G}_{G^{\mathrm{ab}}}$ is the canonical homomorphism attached to $\mu^{\mathrm{ab}} =_{df} (\mathbb{G}_m \xrightarrow{\mu_x} G \to G^{\mathrm{ab}})$ (see 3.34).*

Proof. See [Langlands and Rapoport 1987, 5.3]. □

Remark 4.7.

(a) If $\mathrm{Ker}^1(\mathbb{Q}, G^{\mathrm{ab}}) = 0$, then the condition (d) can be omitted—it is implied by the remaining conditions.

(b) It follows from (a) of the theorem and (A.19) that $I_{\varphi,\mathbb{R}}$ is anisotropic modulo its centre.

The case of a nonsimply connected derived group. In their paper, Langlands and Rapoport define a homomorphism $\varphi\colon \mathfrak{P} \to \mathfrak{G}_G$ to be admissible if it satisfies the conditions of (4.6)—in order not to confuse it with our condition we shall call such a homomorphism is ***LR-admissible***. Their conjecture (ibid. 5.e, p169) states that there is an isomorphism (4.4.1) when the disjoint union is taken over the isomorphism classes of LR-admissible homomorphisms; thus it agrees with (4.4) when G^{der} is simply connected. They then give an example (ibid. pp208–214) that shows that, if their conjecture is true for Shimura varieties with simply connected derived group, then it can not be true for all Shimura varieties. Here we turn their argument around to obtain the opposite conclusion for our version of the conjecture.

In fact we shall need to consider a slight strengthening of (4.4). Let $Z = Z(G)$; then $Z(\mathbb{Q}_p)$ acts on both $\mathrm{Sh}_p(G, X)$ and on $S(\varphi)$, and it commutes with the actions of $G(\mathbb{A}_f^p)$ and the Frobenius element.

Conjecture 4.8. *For any Shimura variety $Sh(G, X)$ and hyperspecial subgroup K_p of $G(\mathbb{Q}_p)$, there exists an isomorphism (4.4.1) commuting with the actions of $Z(\mathbb{Q}_p)$.*

Now consider a Shimura variety $\mathrm{Sh}(G, X)$. Let A be an algebraic subgroup of $Z(G)$ with the property that $H^1(k, A) = 0$ for all fields $k \supset \mathbb{Q}$, and let $G' = G/A$. Write α for the quotient map $G \to G'$ and let X' be the $G'(\mathbb{R})$-conjugacy class of maps $\mathbb{S} \to G'_{\mathbb{R}}$ containing $h_x \circ \alpha$ for all $x \in X$. Then (G', X') defines a Shimura variety.

Theorem 4.9. *Let K_p and K'_p be hyperspecial subgroups of $G(\mathbb{Q}_p)$ and $G'(\mathbb{Q}_p)$ respectively such that $\alpha(K_p) \subset K'_p$. If conjecture (4.8) is true for $Sh_p(G, X)$, then it is also true for $Sh_p(G', X')$.*

The proof will require several lemmas.

Lemma 4.10. *Let $\alpha\colon G \to G'$ be a surjective homomorphism of algebraic groups, and assume that G is connected. For any torus T in G, α defines a surjection from the centralizer $Z_G(T)$ of T in G onto $Z_{G'}(\alpha(T))$. If in addition* $\mathrm{Ker}(\alpha) \subset Z(G)$, *then $Z_G(T) = \alpha^{-1}(Z_{G'}(\alpha(T)))$.*

Proof. The first statement is proved in [Borel 1991, p153]. Assume $\mathrm{Ker}\,\alpha \subset Z(G)$, and suppose $\alpha(g) \in Z_{G'}(\alpha(T))$. Then there exists a $g' \in Z_G(T)$ such that $\alpha(g) = \alpha(g')$, and so $g \in Z_G(T) \cdot \mathrm{Ker}(\alpha) \subset Z_G(T)$.

Lemma 4.11. *Under the hypotheses of the theorem, α maps X bijectively onto X'.*

Proof. The surjectivity of $X \to X'$ follows from the surjectivity of $G'(\mathbb{R}) \to G(\mathbb{R})$. Let M be the $G(\mathbb{C})$-conjugacy class of homomorphisms $\mathbb{G}_m \to G_{\mathbb{C}}$ containing μ_x for each $x \in X$, and let M' be the similar set attached to X'. Because $x \mapsto \mu_x$ is injective, it suffices to show that the map $M \to M'$ defined by α is injective. Fix an x_0, and let x'_0 be its image in X'. Then $M = G(\mathbb{C}) \cdot \mu_{x_0}$ and $M' = G'(\mathbb{C}) \cdot \mu_{x'_0}$, and the injectivity follows from (4.10) applied to $T = \mu_{x_0}(\mathbb{G}_m)$. □

Lemma 4.12. *Under the hypotheses of the theorem, α maps the set of special points of X bijectively onto the set of special points of X'.*

Proof. It is clear that $x \in X$ is special if and only if its image in X' is special. □

Let K^p and K'^p be compact open subgroups of $G(\mathbb{A}_f^p)$ and $G'(\mathbb{A}_f^p)$ respectively such that $\alpha(K^p) \subset K'^p$, and let $K = K^p \cdot K_p$ and $K' = K'^p \cdot K'_p$. It follows from [Tits 1979, 3.9.1] that $\alpha(K)$ is a subgroup of finite index in K', and so, after possibly replacing K with a smaller group, we may assume that $\alpha(K)$ is normal in K'. Then

$$C =_{df} A(\mathbb{Q})\backslash\alpha^{-1}(K')/K$$

is a finite group, which acts on $\mathrm{Sh}(G, X)/K$ on the right. Note that $\alpha\colon K_p \to K'_p$ is surjective, and so $\alpha^{-1}(K') \subset G(\mathbb{A}_f^p) \cdot Z(\mathbb{Q}_p)$, $Z = Z(G)$.

Lemma 4.13. *Over $\mathbb{C}$, the map $Sh(G,X)/K \to Sh(G',X')/K'$ defined by α identifies the second scheme with the quotient of the first under the action of C.*

Proof. This follows easily from the facts that $G(\mathbb{Q}) \to G'(\mathbb{Q})$ and $G(\mathbb{A}_f) \to G'(\mathbb{A}_f)$ are surjective and $X \to X'$ is bijective. □

Lemma 4.14. *For K^p and K'^p sufficiently small, the map*

$$Sh_p(\mathbb{F})/K^p \to Sh'_p(\mathbb{F})/K'^p$$

defined by α identifies the second group with the quotient of the first under the action of C.

Proof. For K^p and K'^p sufficiently small, the map

$$\mathrm{Sh}_p(G,X)/K^p \to \mathrm{Sh}_p(G',X')/K'^p$$

will be étale. Because of our definition (2.9), this statement extends to the integral canonical models, and so the map in the statement of the lemma is surjective. The remainder of the assertion follows from (4.13). □

Lemma 4.15. *Let $\varphi\colon \mathfrak{P} \to \mathfrak{G}_G$ be a special homomorphism, and let $\varphi' = \alpha \circ \varphi$. For K^p and K'^p sufficiently small, the map*

$$S(\varphi)/K^p \to S(\varphi')/K'^p$$

defined by α identifies the second group with the quotient of the first under the action of C.

Proof. The kernel of $I_\varphi \to I_{\varphi'}$ is A, and so the map $I_\varphi(\mathbb{Q}) \to I_{\varphi'}(\mathbb{Q})$ is surjective. The rest of the proof is straightforward. □

After (4.14) and (4.15), in order to prove the theorem, it remains to show that the map $\varphi \mapsto \varphi' =_{df} \alpha \circ \varphi$ defines a one-to-one correspondence between the sets of isomorphism classes of special homomorphisms $\mathfrak{P} \to \mathfrak{G}_G$ and $\mathfrak{P} \to \mathfrak{G}_{G'}$

Lemma 4.16. *Let f_1 and f_2 be homomorphisms $P \to G$ such that $f'_1 =_{df} \alpha \circ f_1$ and $f'_2 =_{df} \alpha \circ f_2$ differ by an inner automorphism of G' and $f_1^{\mathrm{ab}} = f_2^{\mathrm{ab}}$. Then f_1 and f_2 differ by an inner automorphism of G.*

Proof. Since $G^{\mathrm{ad}} = G'^{\mathrm{ad}}$, we can replace f_1 by its composite with an inner automorphism of G to achieve $f'_1 = f'_2$. Now $f_2 = f_1 \cdot \varepsilon$ where ε is a homomorphism $P \to A$. The condition $f_1^{\mathrm{ab}} = f_2^{\mathrm{ab}}$ implies ε maps into $A \cap G^{\mathrm{der}}$, which is a finite group. As P is connected, this implies that $\varepsilon = 1$.

Lemma 4.17. *Let φ_1 and φ_2 be homomorphisms $\mathfrak{P} \to \mathfrak{G}_G$ such that $\varphi_1' \approx \varphi_2'$ and $\varphi_1^{\mathrm{ab}} \approx \varphi_2^{\mathrm{ab}}$. Then $\varphi_1 \approx \varphi_2$.*

Proof. From the preceding lemma, we can suppose that $\varphi_1^{\Delta} = \varphi_2^{\Delta}$. Consider $Isom(\varphi_1, \varphi_2)$. After (A.18) it is a torsor for $Aut(\varphi_1)$, and so defines an element in $H^1(\mathbb{Q}, Aut(\varphi_1))$. We have an exact sequence

$$1 \to A \to Aut(\varphi_1) \to Aut(\varphi_1') \to 1.$$

We are given that the cohomology class of $Isom(\varphi_1, \varphi_2)$ becomes trivial in $H^1(\mathbb{Q}, Aut(\varphi_1'))$, and this implies that it is trivial in $H^1(\mathbb{Q}, Aut(\varphi_1))$ because of $H^1(\mathbb{Q}, A) = 0$. □

Lemma 4.18. *The map $\varphi \mapsto \varphi' =_{df} \alpha \circ \varphi$ is a bijection from the set of isomorphism classes of special homomorphisms $\mathfrak{P} \to \mathfrak{G}_G$ to the set of isomorphism classes of special homomorphisms $\mathfrak{P} \to \mathfrak{G}_{G'}$.*

Proof. It remains to show that, for special points x_1 and x_2 of X,

$$\varphi_{x_1}' \approx \varphi_{x_2}' \Rightarrow \varphi_{x_1} \approx \varphi_{x_2}.$$

But $(\varphi_{x_1})^{\mathrm{ab}} \approx (\varphi_{x_2})^{\mathrm{ab}}$, and so this follows from the last lemma. □

Corollary 4.19. *In order to prove Conjecture (4.8) for all Shimura varieties, it suffices to prove it for those defined by groups with simply connected derived group.*

Proof. Consider a Shimura variety $\mathrm{Sh}(G, X)$ and a hyperspecial subgroup K_p. According to [Milne and Shih, 1982b, 3.4.] there exists a Shimura variety $\mathrm{Sh}(G_1, X_1)$ and a map $\alpha\colon G_1 \to G$ whose kernel A satisfies the above condition and such that G_1^{der} is simply connected and $E(G_1, X_1) = E(G, X)$. Moreover (ibid. 3.1), because G is unramified over $\mathbb{Q}_p$, we can choose α so that $\operatorname{Ker} \alpha$ splits over an unramified extension of $\mathbb{Q}_p$. There exists a hyperspecial subgroup K_p' of $G'(\mathbb{Q}_p)$ such that $\alpha(K_p') \subset K_p$, and we can apply the theorem.

Remark 4.20. Langlands and Rapoport (ibid. §7) construct a map $\alpha\colon G \to G'$ satisfying the conditions (4.8) and an admissible homomorphism $\varphi\colon \mathfrak{P} \to \mathfrak{G}_G$. They construct a cocycle $Z^1(\mathbb{Q}, Aut(\varphi))$ that maps to zero in $H^1(\mathbb{Q}, G'^{\mathrm{ab}})$ and in $H^1(\mathbb{Q}_\ell, G)$ for all ℓ, but not in $H^1(\mathbb{Q}, G^{\mathrm{ab}})$. Note that the kernel of $G^{\mathrm{ab}} \to G'^{\mathrm{ab}}$ is $A/\operatorname{Ker}(G^{\mathrm{der}} \to G'^{\mathrm{der}})$, which need not have trivial cohomology, and so this is not impossible. Now $z \cdot \varphi$ is not LR-admissible (it fails 4.6d) but $z \cdot \varphi'$ is LR-admissible, which shows that their conjecture can not be true for both $\mathrm{Sh}_p(G, X)$ and $\mathrm{Sh}_p(G', X')$. In our terminology, neither is admissible. (If $z \cdot \varphi'$ were admissible, it would be isomorphic to $\varphi_{x'}$ for some special x'; let x be a special point mapping to x'; then $\varphi_x = z\cdot\varphi$ because $(\varphi_x)' = z\cdot\varphi'$, and this contradicts the fact that $(\varphi_x)^{\mathrm{ab}} = \varphi^{\mathrm{ab}}$.)

Remark 4.21. Although the condition that φ be special in (4.4) appears to be the correct one when $\mathrm{Sh}(G,X)$ has good reduction, according to the second example in [Langlands and Rapoport 1987], §7, it is not the correct condition to take in the case of bad reduction. A better understanding of the correct condition for admissibility is needed.

Notes. Apart from the changes noted in the text, this section follows [Langlands and Rapoport 1987]. For a more detailed discussion of the philosophy underlying the conjecture, see [Milne 1991c].

5. THE POINTS OF $\mathrm{Sh}_K(G,X)$ WITH COORDINATES IN $\mathbb{F}_q$

In §4 we gave a conjectural description of the set $\mathrm{Sh}_p(G,X)(\mathbb{F})$ together with the actions of $G(\mathbb{A}_f^p)$ and the Frobenius endomorphism on it. In this section we restate the conjecture in terms of the points with coordinates in $\mathbb{F}_q$.

We retain the notations and assumptions of the first paragraph of §4. In particular, K_p is a hyperspecial subgroup of $G(\mathbb{Q}_p)$ and $\mathrm{Sh}_p(G,X)_v$ is the canonical model of $\mathrm{Sh}_p(G,X) =_{df} \mathrm{Sh}(G,X)/K_p$ over $\mathcal{O}_v$. For simplicity, we assume that the largest $\mathbb{R}$-split subtorus of $Z(G)$ is already split over $\mathbb{Q}$. This implies that $Z(G)(\mathbb{Q})$ is discrete in $Z(G)(\mathbb{A}_f)$ (and in $Z(G)(\mathbb{A}_f^p)$).

In general the scheme $\mathrm{Sh}_p(G,X)_v$ has no points with coordinates in a finite field. Thus in order to obtain a nonvacuous statement, we choose a compact open subgroup K^p of $G(\mathbb{A}_f^p)$, write $K = K^p\cdot K_p$, and define

$$\mathrm{Sh}_K(G,X) = \mathrm{Sh}(G,X)/(K^p\cdot K_p) = \mathrm{Sh}_p(G,X)/K^p.$$

Since $\mathrm{Sh}_p(G,X)$, together with the action of $G(\mathbb{A}_f^p)$, is assumed to have a canonical model over $\mathcal{O}_v$, we can regard $\mathrm{Sh}_K(G,X)$ as being defined over $\mathcal{O}_v$. Thus $\mathrm{Sh}_K(\mathbb{F}_q) =_{df} \mathrm{Sh}_K(G,X)(\mathbb{F}_q)$ is defined for every field $\mathbb{F}_q$ containing $\kappa(v)$.

Statement of the conjecture. A motive M over $\mathbb{F}_q$ gives a motive $M_{\mathbb{F}}$ over $\mathbb{F}$ together with a Frobenius element $\epsilon\in\mathrm{Aut}(M_{\mathbb{F}})$, and the pair $(M_{\mathbb{F}},\epsilon)$ should determine the isomorphism class of M over $\mathbb{F}_q$. This suggests that, in describing $\mathrm{Sh}_K(\mathbb{F}_q)$, we should consider pairs (φ,ϵ) where φ is an admissible homomorphism $\mathfrak{P}\to\mathfrak{G}_G$ and ϵ is an element of $I_\varphi(\mathbb{Q})$, i.e., ε is an automorphism of φ. For the moment, we impose no condition on ϵ. If (φ,ϵ) and (φ',ϵ') are two such pairs, then we define

$$\begin{aligned}\mathrm{Isom}((\varphi,\epsilon),(\varphi',\epsilon')) &= \{\alpha \in \mathrm{Isom}(\varphi,\varphi') \mid \alpha(\epsilon)=\epsilon'\}\\ &= \{g \in G(\mathbb{Q}^{\mathrm{al}}) \mid \mathrm{ad}(g)\circ\varphi = \varphi', \quad \mathrm{ad}(g)(\epsilon)=\epsilon'\}.\end{aligned}$$

Also, we set

$$\begin{aligned} I_{\varphi,\epsilon}(\mathbb{Q}) &= \mathrm{Aut}(\varphi,\epsilon) = \text{centralizer of } \epsilon \text{ in } I_\varphi(\mathbb{Q}),\\ X^p(\varphi,\epsilon) &= \{x^p \in X^p(\varphi) \mid \epsilon x^p \equiv x^p \bmod K^p\},\\ X_p(\varphi,\epsilon) &= \{x_p \in X_p(\varphi) \mid \epsilon x_p = \Phi^r x_p\},\end{aligned}$$

where $r = [\mathbb{F}_q \colon \kappa(v)]$. Finally, we define

$$S_K(\varphi,\epsilon) = I_{\varphi,\epsilon}(\mathbb{Q})\backslash X^p(\varphi,\epsilon)\times X_p(\varphi,\epsilon)/K^p.$$

Note that, for any $g \in G(\mathbb{A}_f^p)$ and compact open subgroups K^p and K'^p of $G(\mathbb{A}_f^p)$ such that $K'^p \subset gK^pg^{-1}$, there is a map

$$S_{K'}(\varphi,\epsilon) \to S_K(\varphi,\epsilon), \quad [x^p,x_p]\mapsto [x^p\cdot g, x_p].$$

Conjecture 5.1. *There is a family of bijections*

$$\alpha_K\colon Sh_K(\mathbb{F}_q) \longrightarrow \textstyle\coprod_{\varphi,\varepsilon} S_K(\varphi,\varepsilon),$$

one for each sufficiently small compact open subgroup K^p in $G(\mathbb{A}_f^p)$, such that the diagram

$$\begin{array}{ccc} Sh_{K'}(\mathbb{F}_q) & \xrightarrow{\alpha_{K'}} & \coprod_{\varphi,\epsilon} S_{K'}(\varphi,\epsilon)\\ \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle g}\\ Sh_K(\mathbb{F}_q) & \xrightarrow{\alpha_K} & \coprod_{\varphi,\epsilon} S_K(\varphi,\epsilon)\end{array}$$

commutes for any $g \in G(\mathbb{A}_f^p)$ such that $K' \subset gKg^{-1}$. The disjoint unions are over a set of representatives for the isomorphism classes of pairs (φ,ϵ) with φ admissible and $\epsilon \in I_\varphi(\mathbb{Q})$.

Remark 5.2. Two pairs (φ,ϵ) and (φ,ϵ') are isomorphic if and only if there is an element $g \in G(\mathbb{Q}^{\mathrm{al}})$ such that $\mathrm{ad}(g)\circ\varphi=\varphi$ and $g\cdot\epsilon\cdot g^{-1}=\epsilon'$. The first condition on g implies that it is in $I_\varphi(\mathbb{Q})$. Therefore, for a fixed φ, the isomorphism classes of pairs (φ,ϵ) are parametrized by the conjugacy classes in $I_\varphi(\mathbb{Q})$.

In the remainder of this section, we prove that for a given Shimura variety $\mathrm{Sh}(G,X)$, hyperspecial group K_p, and good prime v of $E(G,X)$, this conjecture is implied by the Main Conjecture 4.4.

A combinatorial lemma. The proof is based on the following easy combinatorial lemma.

Lemma 5.3. *Let I be a group, and let X and Y be left I-sets. Let a and b be maps of I-sets $Y \to X$. Suppose that there is a subgroup C of the centre of I with the following properties:*

(i) *the isotropy group in I at every $x \in a(Y)$ is C;*
(ii) *if $ghg^{-1}h^{-1} \in C$, $g, h \in I$, then $ghg^{-1}h^{-1} = 1$, i.e., $I^{\mathrm{der}} \cap C = 1$.*

Then the set $(I\backslash Y)^{a=b}$ on which the maps $I\backslash Y \to I\backslash X$ defined by a and b agree is a disjoint union

$$(I\backslash Y)^{a=b} = \textstyle\coprod_h I_h\backslash Y_h$$

where
$Y_h = \{y \in Y \mid hay = by\}$,
$I_h = \{g \in I \mid gh = hg\}$, *i.e., I_h is the centralizer of h in I,*
and h runs through a set of representatives in I for the conjugacy classes of I/C.

Proof. Clearly I_h does act on Y_h, and the inclusion map $Y_h \to Y$ gives a well-defined map $I_h\backslash Y_h \to I\backslash Y$ whose image is contained in $(I\backslash Y)^{a=b}$. Let y represent an element of $(I\backslash Y)^{a=b}$. Then $gay = by$ for some $g \in I$, and, because the h's are a set of representatives, $g = ihi^{-1}c$ for some $i \in I$, $c \in C$, and some h. Now, using (i), one finds that $i^{-1}y \in Y_h$. Since $i^{-1}y$ represents the same class as y, this shows that y is in the image of the map

$$\textstyle\coprod_h I_h\backslash Y_h \to (I\backslash Y)^{a=b},$$

which is therefore surjective. It remains to show that this map is injective. Let y and y' be elements of Y_h and $Y_{h'}$ respectively that represent the same class in $I\backslash Y$. We are given that:

$$hay = by, \quad h'ay' = by', \quad y' = gy \text{ some } g \in I.$$

On substituting from the last equation into the second, we find that

$$h'agy = bgy, \text{ or } h'gay = gby.$$

On comparing this with the first equation, we see that h and $g^{-1}h'g$ have the same action on ay, and it follows from (i) that the two elements differ by an element $c \in C$,

$$g^{-1}h'g = hc.$$

This shows that h and h' lie in the same conjugacy class of I/C, and because the h's form a set of representatives for these classes, this means $h = h'$. Now the equation $g^{-1}hg = hc$ and assumption (ii) imply that $h = g^{-1}hg$, i.e., that $g \in I_h$, and so y and y' define the same class in $I_h \backslash Y_h$. □

Remark 5.4. In the statement of the lemma, we have allowed h to run through a complete set of representatives for the conjugacy classes in I/C, but clearly we need only take those h for which there is a $y \in Y$ such that $hay = by$.

The Main Conjecture implies 5.1. Before we can apply (5.3), we need a lemma.

Lemma 5.5. *Let Z denote the centre of G, and let φ be an admissible homomorphism $\mathfrak{P} \to \mathfrak{G}_G$. For sufficiently small K^p, the following hold:*

(a) *if $\varepsilon \in I_\varphi(\mathbb{Q})$ and $\varepsilon x = x$ for some $x \in (X^p/K^p) \times X_p$, then $\varepsilon \in Z(\mathbb{Q}) \cap K$;*

(b) *$I_\varphi(\mathbb{Q})^{\mathrm{der}} \cap Z(\mathbb{Q}) \cap K = \{1\}$.*

Proof. (a) After dividing by Z, we can assume $Z = 1$. Since $I_\varphi(\mathbb{R})$ is compact (see 4.7b), ε is semisimple, and so lies in a subtorus T of G. I claim that if χ is a rational character of T and v is a valuation of $\mathbb{Q}^{\mathrm{al}}$, then $|\chi(\varepsilon)|_v = 1$. If v is archimedean, this is a consequence of the compactness of $I_\varphi(\mathbb{R})$. If $v|p$, it is a consequence of the equation $\varepsilon x_p = x_p$. If v is nonarchimedean but prime to p, the equation $\varepsilon x^p = x^p$ implies ε is conjugate to an element of K^p, which implies the claim. We conclude that $\chi(\varepsilon)$ is a root of unity. Since $\chi(\varepsilon)$ lies in a Galois extension whose degree is at most the product of the order of the Weyl group of G with the order of the group of automorphisms of the Dynkin diagram, it is one of a fixed finite set of roots of unity. We have merely to take K^p sufficiently small that the resulting congruence conditions force it to be 1.

(b) Clearly $I_\varphi(\mathbb{Q})^{\mathrm{der}} \cap Z(\mathbb{Q})$ is contained in the centre of I_φ^{der}, which is finite. □

Proposition 5.6. *If the Main Conjecture 4.4 is true for $Sh_p(G, X)$, then so also is Conjecture 5.1.*

Proof. Fix a bijection

$$(\mathrm{Sh}_p(\mathbb{F}), \Phi, \times) \to \coprod\nolimits_\varphi (S(\varphi), \Phi(\varphi), \times(\varphi)).$$

On dividing out by K^p, we obtain a bijection

$$\mathrm{Sh}_K(\mathbb{F}) \to \coprod\nolimits_\varphi S_K(\varphi), \tag{5.6.1}$$

where $S_K(\varphi) =_{df} S(\varphi)/K^p$.

If K^p is sufficiently small that Lemma 5.5 holds, then we can apply Lemma 5.3 with $Y = (X^p \times X_p)/K^p = X$, $I = I_\varphi(\mathbb{Q})$, $C = Z(\mathbb{Q}) \cap K$ (note that Z is a subgroup of the centre of I_φ), $a = \mathrm{id}$, and $b = \Phi^r$ where $r = [\mathbb{F}_q \colon \kappa(v)]$. This provides us with a bijection

$$S(\varphi)^{\Phi^r=1} \to \coprod_\epsilon I_{\varphi,\epsilon}(\mathbb{Q})\backslash X^p(\varphi,\epsilon) \times X_p(\varphi,\epsilon).$$

Here ϵ runs over a set of representatives for the conjugacy classes in $I_\varphi(\mathbb{Q})/C$. Because of the assumption on $Z(G)$ in the second paragraph of this section, when K is sufficiently small, $C = 1$. On combining these bijections for the different φ's with the bijection $(*)$, we obtain a bijection

$$\mathrm{Sh}_K(\mathbb{F}_q) \to \coprod_{\varphi,\epsilon} I_{\varphi,\epsilon}(\mathbb{Q})\backslash X^p(\varphi,\epsilon) \times X_p(\varphi,\epsilon)/K^p.$$

As K varies, these give the commutative diagrams required by Conjecture 5.1.

Remark 5.7. One obtains a similar statement to (5.1) without the assumption on Z in the second paragraph, but it is more complicated to state.

Definition 5.8. A pair (φ,ε) is said to be ***admissible*** if there exists an $x_p \in X_p(\varphi)$ such that $\varepsilon x_p = \Phi^r x_p$.

Example 5.9. Consider the case of a Shimura variety $\mathrm{Sh}(T,x)$ defined by a torus. Then (φ,ε) is admissible if and only if $\varphi = \varphi_x$ and the image of ε in $X_*(T)^{\mathrm{Gal}(\mathbb{Q}_p^{\mathrm{al}}/\mathbb{Q}_p)}$ is equal to $\mathrm{Tr}_{B(\mathbb{F}_q)/\mathbb{Q}_p}\, \mu_x$.

Notes. Lemma 5.3 was suggested by [Kottwitz 1984b], §1. The proof of Lemma 5.5 is taken from [Langlands 1979 , p1171].

6. INTEGRAL FORMULAS

In this section, we derive from the Main Conjecture 4.4 an expression for $\sum_{t'} \mathrm{Tr}(\mathcal{T}(g)^{(r)}|\mathcal{V}_t(\xi))$ as a sum of products of (twisted) orbital integrals.

We retain the notations and assumptions of the first two paragraphs of §4. For simplicity, we also assume that the largest $\mathbb{R}$-split subtorus of $Z(G)$ is already split over $\mathbb{Q}$, and that G^{der} is simply connected.

Triples. Fix a field $\mathbb{F}_q$ of degree r over $\kappa(v)$. We wish to consider triples $(\gamma_0;\gamma,\delta)$ where

6.1.1. γ_0 is a semisimple element of $G(\mathbb{Q})$ that is elliptic in $G(\mathbb{R})$ (i.e., contained in an elliptic torus in $G_\mathbb{R}$);

6.1.2. $\gamma = (\gamma(\ell))_{\ell \neq p,\infty}$ is an element of $G(\mathbb{A}_f^p)$ such that, for all ℓ, $\gamma(\ell)$ becomes conjugate to γ_0 in $G(\mathbb{Q}_\ell^{\mathrm{al}})$;

6.1.3. δ is an element of $G(B(\mathbb{F}_q))$ such that

$$\mathcal{N}\delta =_{df} \delta \cdot \sigma\delta \cdot \ldots \cdot \sigma^{n-1}\delta, \quad n = [\mathbb{F}_q \colon \mathbb{F}_p],$$

becomes conjugate to γ_0 in $G(\mathbb{Q}_p^{\mathrm{al}})$;

6.1.4. for any special $x \in X$, the image of δ under the map $\mathbb{B}(G) \to \pi_1(G)_{\Gamma(p)}$ of (B.27) is the class of $-\mu_x$.

Given such a triple $(\gamma_0; \gamma, \delta)$, we set

$I_0 = G_{\gamma_0}$, the centralizer of γ_0 in G; because γ_0 is semisimple and G^{der} is simply connected, I_0 is connected and reductive.

$I_\infty =$ the inner form of $I_{0\mathbb{R}}$ such that $I_\infty/Z(G)$ is anisotropic; more precisely, choose an elliptic maximal torus T of $G_\mathbb{R}$ containing γ_0, and let x be an element of X such that h_x factors through T; then $\operatorname{ad} h(i)$ preserves $I_{0\mathbb{R}}$ and induces a Cartan involution on $(I_0/Z(G))_\mathbb{R}$, which we use to twist $I_{0\mathbb{R}}$.

$I_\ell =$ the centralizer of $\gamma(\ell)$ in $G_{\mathbb{Q}_\ell}$;

$I_p =$ the inner form of $G_{\mathbb{Q}_p}$ such that $I_p = \{x \in G(B(\mathbb{F}_q)) \mid x^{-1} \cdot \delta \cdot \sigma x = \delta\}$.

We need to make another assumption about the triple:

(6.1.5) There exists an inner form (I, α) of I_0 such that $I_{\mathbb{Q}_\ell}$ is isomorphic to I_ℓ for all ℓ (including p and ∞).

Remark 6.2. For each ℓ, I_ℓ is an inner form of $I_{0,\mathbb{Q}_\ell}$ and hence defines a cohomology class $e(\ell) \in H^1(\mathbb{Q}_\ell, I_0^{\mathrm{ad}})$. There will exist an inner form I of I_0 as above if and only if there is a class $e_0 \in H^1(\mathbb{Q}, I_0^{\mathrm{ad}})$ whose image in $\prod_\ell H^1(\mathbb{Q}_\ell, I_0^{\mathrm{ad}})$ is $e(\ell)$ for each ℓ. From the exact sequence (B.24),

$$0 \to H^1(\mathbb{Q}, I_0^{\mathrm{ad}}) \to \oplus_\ell H^1(\mathbb{Q}_\ell, I_0^{\mathrm{ad}}) \to A(I_0^{\mathrm{ad}})$$

we see that e_0 exists if and only if $(e(\ell))$ maps to zero in $A(I_0^{\mathrm{ad}})$, and that, in this case, it is unique. This gives a criterion for the existence of (I, α), and shows that it is unique up to a nonunique inner automorphism when it exists.

Because γ_0 and γ_ℓ are stably conjugate, there exists an isomorphism $a_\ell \colon I_{0,\mathbb{Q}_\ell^{\mathrm{al}}} \to I_{\ell,\mathbb{Q}_\ell^{\mathrm{al}}}$, well-defined up to an inner automorphism of I_0 over $\mathbb{Q}_\ell^{\mathrm{al}}$. Choose a system $(I, a, (j_\ell))$ consisting of a $\mathbb{Q}$-group I, an inner twisting $a \colon I_0 \to I$ (isomorphism over $\mathbb{Q}^{\mathrm{al}}$), and isomorphisms $j_\ell \colon I_{\mathbb{Q}_\ell} \to I_\ell$ over $\mathbb{Q}_\ell$ for all ℓ, unramified for almost all ℓ, such that $j_\ell \circ a$ and a_ℓ differ by an inner automorphism—our assumption

(6.1.5) guarantees the existence of such a system. Moreover, any other such system is isomorphic to one of the form $(I, a, (j_\ell \circ \operatorname{ad} h_\ell))$ where $(h_\ell) \in I^{\mathrm{ad}}(\mathbb{A})$.

Let dx denote the Haar measure on $G(\mathbb{A}_f^p)$ giving measure 1 to K^p. Choose a Haar measure di^p on $I(\mathbb{A}_f^p)$ that gives rational measure to compact open subgroups of $I(\mathbb{A}_f^p)$, and use the isomorphisms j_ℓ to transport it to a measure on $G(\mathbb{A}_f^p)_\gamma$ (the centralizer of γ in $G(\mathbb{A}_f^p)$). The resulting measure does not change if (j_ℓ) is modified by an element of $I^{\mathrm{ad}}(\mathbb{A})$. Write $d\bar{x}$ for the quotient of dx by di^p. Let f be an element of the Hecke algebra $\mathcal{H}$ of locally constant K-bi-invariant $\mathbb{Q}$-valued functions on $G(\mathbb{A}_f)$, and assume that $f = f^p \cdot f_p$ where f^p is a function on $G(\mathbb{A}_f^p)$ and f_p is the characteristic function of K_p in $G(\mathbb{Q}_p)$ divided by the measure of K_p. Define

$$O_\gamma(f^p) = \int_{G(\mathbb{A}_f^p)_\gamma \backslash G(\mathbb{A}_f^p)} f^p(x^{-1}\gamma x)\, d\bar{x} \tag{6.3.1}$$

Let dy denote the Haar measure on $G(B(\mathbb{F}_q))$ giving measure 1 to $G(W(\mathbb{F}_q))$. Choose a Haar measure di_p on $I(\mathbb{Q}_p)$ that gives rational measure to the compact open subgroups, and use j_p to transport the measure to $I_p(\mathbb{Q}_p)$. Again the resulting measure does not change if j_p is modified by an element of $I^{\mathrm{ad}}(\mathbb{Q}_p)$. Write $d\bar{y}$ for the quotient of dy by di_p. Let S be a maximal $B(\mathbb{F}_q)$-split subtorus of $G_{B(\mathbb{F}_q)}$ whose apartment contains the hyperspecial point fixed by K_p. The conjugacy class $\mathfrak{c}(X)_{\mathbb{Q}_p^{\mathrm{al}}}$ is defined over $E_v \subset B(\mathbb{F}_q)$, and (1.7) shows that it is represented by a cocharacter μ of S defined over $B(\mathbb{F}_q)$ and moreover that the coset

$$G(W(\mathbb{F}_q)) \cdot \mu(p) \cdot G(W(\mathbb{F}_q))$$

is independent of all choices. Let ϕ_r be the characteristic function of this coset, and define

$$TO_\delta(\phi_r) = \int_{I(\mathbb{Q}_p)\backslash G(B(\mathbb{F}_q))} \phi_r(y^{-1}\delta\sigma(y))d\bar{y} \tag{6.3.2}$$

Since $I/Z(G)$ is anisotropic over $\mathbb{R}$, and since we are assuming that the largest subtorus of $Z(G)$ split over $\mathbb{R}$ is already split over $\mathbb{Q}$, $I(\mathbb{Q})$ is a discrete subgroup of $I(\mathbb{A}_f^p)$, and we can define the volume of $I(\mathbb{Q})\backslash I(\mathbb{A}_f)$. It is a rational number because of our assumption on di^p and di_p. Finally, define

$$I(\gamma_0;\gamma,\delta) = I(\gamma_0;\gamma,\delta)(f^p, r) = \operatorname{vol}(I(\mathbb{Q})\backslash I(\mathbb{A}_f)) \cdot O_\gamma(f^p) \cdot TO_\delta(\phi_r) \tag{6.3.3}$$

Proposition 6.4. *Let*

$$Y^p = \{g \in G(\mathbb{A}_f^p)/K^p \mid \gamma g \equiv g \pmod{K)^p}\}$$
$$Y_p = \{g \in G(B(\mathbb{F}_q)/W(\mathbb{F}_q)) \mid g^{-1}\cdot\delta\cdot\sigma g \in G(W(\mathbb{F}_q))\cdot\mu(p)\cdot G(W(\mathbb{F}_q))\}.$$

When we take f^p to be the characteristic function of K^p, then

$$\mathrm{Card}(I(\mathbb{Q})\backslash Y^p \times Y_p) = I(\gamma_0;\gamma,\delta)(f^p, r).$$

Proof. We have

$$\begin{aligned}\mathrm{Card}(I(\mathbb{Q})\backslash Y^p \times Y_p) &= \sum_{[g,h]\in I(\mathbb{Q})\backslash G(\mathbb{A}_f^p)\times G(B(\mathbb{F}_q))} f^p(g^{-1}\gamma g)\cdot \phi_r(h^{-1}\delta\sigma h)\\ &= \int_{I(\mathbb{Q})\backslash G(\mathbb{A}_f^p)\times G(B(\mathbb{F}_q))} f^p(g^{-1}\gamma g)\cdot \phi_r(h^{-1}\delta\sigma h)\\ &= \mathrm{vol}(I(\mathbb{Q})\backslash G(\mathbb{A}_f^p)_\gamma \times G_{\delta\sigma}(B(\mathbb{F}_q))\cdot\\ &\qquad \int_{G(\mathbb{A}_f^p)_\gamma\backslash G(\mathbb{A}_f^p)} f^p(g^{-1}\gamma g)\cdot \int_{G_{\delta\sigma}(\mathbb{Q}_p)\backslash G(B(\mathbb{F}_q))} \phi_r(g^{-1}\delta\sigma g).\end{aligned}$$

But $I(\mathbb{A}_f^p) \approx G(\mathbb{A}_f^p)$ and $I(\mathbb{Q}_p) \approx G_{\delta\sigma}(\mathbb{Q}_p)$, and so this proves the formula. □

Definition 6.5. Two triples $(\gamma_0;\gamma,\delta)$ and $(\gamma_0';\gamma',\delta')$ are said to be ***equivalent***, $(\gamma_0;\gamma,\delta) \sim (\gamma_0';\gamma',\delta')$ if γ_0 is conjugate to γ_0' in $G(\mathbb{Q})$, $\gamma(\ell)$ is conjugate to $\gamma'(\ell)$ in $G(\mathbb{Q}_\ell)$ for each $\ell \neq p,\infty$, and δ is σ-conjugate to δ' in $G(B(\mathbb{F}_q))$.

Remark 6.6. The integral $I(\gamma_0;\gamma,\delta)$ is independent of the choices made, and, it is implicit in the work of Kottwitz that, appropriately interpreted,

$$(\gamma_0;\gamma,\delta) \sim (\gamma_0';\gamma',\delta') \implies I(\gamma_0;\gamma,\delta) = I(\gamma_0';\gamma',\delta').$$

The triple attached to an admissible pair (φ,ε). Let $a\colon H \to H^*$ be an inner twist of a group H over k (recall that this means a is defined over k^{al}). We say that a maximal torus T^* of H^* ***comes from*** H if there is a torus $T \subset H$ and a $g \in G^*(k^{\mathrm{al}})$ such that $\mathrm{ad}\, g \circ a$ maps T onto T^* and is defined over k.

Lemma 6.7. *Let $a\colon G \to G^*$ be an inner twist of G. If $T^* \subset G^*$ comes from G everywhere locally and $T^*/Z(G^*)$ is anisotropic for at least one prime of $\mathbb{Q}$, then T^* comes from G globally.*

Proof. See [Langlands and Rapoport 1987 , 5.6.] □

Proposition 6.8. *Any admissible pair (φ,ε) is isomorphic to an admissible pair (φ',ε') with φ' special, say $\varphi' = \varphi_x$, and $\varepsilon' \in T(\mathbb{Q})$ where (T,x) is a special pair in (G,X).*

Proof. Ibid. 5.23. □

Lemma 6.9. *For any special φ, the σ-conjugacy class b_φ is represented by an element $\delta \in G(B(\mathbb{F}_q))$; δ is well-defined up to σ-conjugacy.*

Proof. This is clear from its definition. □

Now consider an admissible pair (φ, ε). After (6.8), we can assume that φ is special and that $\varepsilon \in T(\mathbb{Q})$. Write γ_0 for ε regarded as an element of $G(\mathbb{Q})$, and write γ for the image of ε in $G(\mathbb{A}_f^p)$.

Lemma 6.10. *The triple satisfies the conditions (6.1), and isomorphic pairs give equivalent triples.*

Proof. The conditions (6.1.1–4) follow easily from the definition. The group I can be taken to be $I_{\varphi,\varepsilon}$. □

Thus we have a map $t\colon [\varphi, \varepsilon] \mapsto [\gamma_0; \gamma, \delta]$ from the set of isomorphism classes of admissible pairs to the set of equivalence classes of triples. Call a class $[\gamma_0; \gamma, \delta]$ ***effective*** if it arises from an admissible pair, and a triple $(\gamma_0; \gamma, \delta)$ ***effective*** if it belongs to such a class.

Proposition 6.11. *Fix an effective triple $(\gamma_0; \gamma, \delta)$; then the number of isomorphism classes of admissible pairs (φ, ε) with $t(\varphi, \varepsilon) = [\gamma_0; \gamma, \delta]$ is equal to*

$$c(\gamma_0) =_{df} \operatorname{Card}(\operatorname{Ker}(\operatorname{Ker}^1(\mathbb{Q}, I_0) \to H^1(\mathbb{Q}, G))).$$

Proof. Fix an admissible pair $(\varphi_0, \varepsilon_0)$, and consider a second admissible pair (φ, ε). Then $Hom((\varphi_0, \varepsilon_0), (\varphi, \varepsilon))$ is a $I_{\varphi_0,\varepsilon_0}$-torsor, and hence defines a class in $H^1(\mathbb{Q}, I_{\varphi_0,\varepsilon_0})$. The fact that $(\varphi_0, \varepsilon_0)$ and (φ, ε) define the same triple implies that the class is trivial locally, and it maps to zero in $H^1(\mathbb{Q}, G^{\mathrm{ab}})$ because a torus has only one admissible pair up to equivalence. Thus $(\varphi, \varepsilon) \mapsto Hom((\varphi_0, \varepsilon_0), (\varphi, \varepsilon))$ is a map from the set of admissible pairs with $t(\varphi, \varepsilon) = [\gamma_0; \gamma, \delta]$ into the set

$$\operatorname{Ker}(\operatorname{Ker}^1(\mathbb{Q}, I_{\varphi_0,\varepsilon_0}) \to H^1(\mathbb{Q}, G^{\mathrm{ab}})),$$

and it is easy to see that this is a bijection. An elementary lemma [Langlands and Rapoport 1987, 5.24] shows that the cardinality of this set does not change when $I_{\varphi_0,\varepsilon_0}$ is replaced with I_0, and, because G^{der} satisfies the Hasse principle, G^{ab} can be replaced with G. □

Proposition 6.12. *Let (φ, ε) be an admissible pair, and let $S_K(\varphi, \varepsilon)$ be the set defined in §5. Let $(\gamma_0; \gamma, \delta)$ be the triple associated with (φ, ε), and take f^p to be the characteristic function of K^p. Then*

$$\operatorname{Card} S_K(\varphi, \varepsilon) = I(\gamma_0; \gamma, \delta).$$

Proof. We know from (6.4) that $I(\gamma_0;\gamma,\delta) = \mathrm{Card}(I(\mathbb{Q})\backslash Y^p \times Y_p)$, and it is obvious from the definitions that $I = I_{\varphi,\varepsilon}$ and $Y^p = X^p(\varphi,\varepsilon)$. We leave it as an exercise to show that $Y_p = X_p(\varphi,\varepsilon)$. □

Corollary 6.13. *If the main conjecture 4.4 is true for $Sh_p(G,X)$, then*

$$\mathrm{Card}(Sh_p(\mathbb{F}_q)) = \sum_{(\gamma_0;\gamma,\delta)} c(\gamma_0)\cdot I(\gamma_0;\gamma,\delta)$$

where the sum is over a set of representatives for the effective triples.

Proof. Combine (6.12) with (5.6). □

Addition of a local system. A rational representation $\xi\colon G \to \mathrm{GL}(V)$ of G defines a $\mathbb{Q}_\ell$-local system $\mathcal{V}(\xi)$ on $\mathrm{Sh}_p(G,X)$.

Proposition 6.14. *If the Main Conjecture 4.4 is true for $Sh_p(G,X)$, then*

$$\sum_t \mathrm{Tr}(\Phi^r \mid \mathcal{V}(\xi)_t) = \sum_{(\gamma_0;\gamma,\delta)} c(\gamma_0)\cdot I(\gamma_0;\gamma,\delta)\cdot \mathrm{Tr}\,\xi(\gamma_0)$$

where first sum is over the points of $Sh_p(\mathbb{F}_q)$ and the second sum is over a set of representatives for the equivalence classes of effective triples.

Proof. When ξ is taken to be the trivial representation of G on $V = \mathbb{Q}$, this is just (6.13). To see that the formula is correct, it suffices to check it in the case of a torus, and this is obvious from the definitions. □

Addition of a Hecke operator.

Theorem 6.15. *Assume that (4.4) holds for $Sh_p(G,X)$. Then for any $g \in G(\mathbb{A}_f^p)$,*

$$\textstyle\sum_{t'} \mathrm{Tr}(\mathcal{T}(g)^{(r)}|\mathcal{V}_t(\xi)) = \sum_{(\gamma_0;\gamma,\delta)} c(\gamma_0)\cdot I(\gamma_0;\gamma,\delta)\cdot \mathrm{Tr}\,\xi(\gamma_0),$$

where, in the definition of $I(\gamma_0;\gamma,\delta)$, f^p is taken to be the characteristic function of $K_p\cdot g\cdot K_p$ and the sum is over a set of representatives of the equivalence classes of effective triples

Proof. When $g = 1$, this is (6.14), and the proof in the general case is similar. □

Remark 6.16.

(a) In the statement of (6.15), one can replace g with any f^p as at the start of this section.

(b) In (6.14) and (6.15), one can replace ξ with a representation over a number field.

Notes. The definition of $I(\gamma_0;\gamma,\delta)$ follows [Kottwitz 1990, §3] very closely, sometimes word for word.

7. A CRITERION FOR EFFECTIVENESS

In this section derive a criterion for a triple $(\gamma_0;\gamma,\delta)$ to arise from an admissible pair (φ,ε). As a consequence we find that the formula conjectured by Kottwitz [Kottwitz 1990, 3.1] follows from the Main Conjecture 4.4.

We retain the notations and assumptions of the second paragraph of §6.

Definition of the group $\mathfrak{k}^\vee(I_0/\mathbb{Q})$. Let γ_0 be a semisimple element of $G(\mathbb{Q})$, and let I_0 be the centralizer of γ_0. Because G^{der} is simply connected, I_0 is connected. Let $I_0' = I_0 \cap G^{\text{der}}$. There is an exact sequence

$$1 \to I_0' \to I_0 \to G^{\text{ab}} \to 1.$$

Because G^{der} is simply connected, $\pi_1(G) = \pi_1(G^{\text{ab}})$, and so (see B.2a) there is an exact sequence:

$$1 \to \pi_1(I_0') \to \pi_1(I_0) \to \pi_1(G) \to 1.$$

Recall (Appendix B) that for a group H over $\mathbb{Q}$, $A(H) = \pi_1(H)_{\Gamma,\text{tors}}$ where $\Gamma = \text{Gal}(\mathbb{Q}^{\text{al}}/\mathbb{Q})$. We write $A_\ell(H)$ for $A(H_{\mathbb{Q}_\ell}) =_{df} \pi_1(H)_{\Gamma(\ell),\text{tors}}$, where $\Gamma(\ell) = \text{Gal}(\mathbb{Q}_\ell^{\text{al}}/\mathbb{Q}_\ell)$. An embedding $\mathbb{Q}^{\text{al}} \hookrightarrow \mathbb{Q}_\ell^{\text{al}}$ induces an inclusion $\Gamma(\ell) \hookrightarrow \Gamma$, and hence a map

$$A_\ell(H) \to A(H).$$

In (B.22) we construct a canonical map:

$$\alpha_H\colon H^1(\mathbb{Q}_\ell, H) \to A_\ell(H).$$

On composing the boundary map $G^{\text{ab}}(\mathbb{Q}_\ell) \to H^1(\mathbb{Q}_\ell, I_0')$ with $\alpha_{I_0'}$, we obtain a canonical map

$$G^{\text{ab}}(\mathbb{Q}_\ell) \to A_\ell(I_0').$$

Definition 7.1. The group $\mathfrak{k}^\vee(I_0/\mathbb{Q})$ is the quotient of $A(I_0')$ by the subgroup generated by the images of the maps $G^{\text{ab}}(\mathbb{Q}_\ell) \to A(I_0')$, i.e., it is the group making the sequence

$$\oplus_{\text{all }\ell} G^{\text{ab}}(\mathbb{Q}_\ell) \to A(I_0') \to \mathfrak{k}^\vee(I_0/\mathbb{Q}) \to 1$$

exact.

At the end of this section, we show that $\mathfrak{k}^\vee(I_0/\mathbb{Q})$ is the dual of Kottwitz's group $\mathfrak{k}(I_0/\mathbb{Q})$, which explains our notation.

A criterion for the existence of an admissible homomorphism satisfying local conditions. Let $\varphi\colon \mathfrak{P} \to \mathfrak{G}_G$ be a homomorphism, and let (φ_ℓ) be a family of homomorphisms $\mathfrak{P}(\ell) \to \mathfrak{G}_G(\ell)$. We seek necessary and sufficient conditions for there to exist a homomorphism $\varphi_1\colon \mathfrak{P} \to \mathfrak{G}_G$ such that

(7.2.1) $\varphi_1^\Delta \approx \varphi^\Delta$ (i.e., the restrictions of φ and φ_1 to P are conjugate);

(7.2.2) $\varphi_1(\ell) \approx \varphi_\ell$ for all ℓ;

(7.2.3) $\varphi_1^{\mathrm{ab}} \approx \varphi^{\mathrm{ab}}$ (i.e., the composites of φ and φ_1 with $\mathfrak{G}_G \to \mathfrak{G}_{G^{\mathrm{ab}}}$ are isomorphic).

Obviously, a necessary condition for this is that

(7.3.1) $\varphi(\ell)^\Delta \approx \varphi_\ell^\Delta$;

(7.3.2) $\varphi_\ell^{\mathrm{ab}} \approx \varphi(\ell)^{\mathrm{ab}}$.

Henceforth we assume these two conditions. Let $I = \mathrm{Aut}(\varphi)$. The composite of $I_{\mathbb{Q}^{\mathrm{al}}} \to G_{\mathbb{Q}^{\mathrm{al}}} \to G^{\mathrm{ab}}_{\mathbb{Q}^{\mathrm{al}}}$ is surjective, and is defined over $\mathbb{Q}$. We let I' be its kernel, so that the sequence

$$1 \to I' \to I \to G^{\mathrm{ab}} \to 1$$

is exact, and we define $\mathfrak{k}^\vee(I/\mathbb{Q})$ as above. Consider the exact commutative diagram,

$$\begin{array}{ccccc}
 & & \oplus_\ell G^{\mathrm{ab}}(\mathbb{Q}_\ell) & & \\
 & & \downarrow & & \\
H^1(\mathbb{Q}, I') & \longrightarrow & \oplus_\ell H^1(\mathbb{Q}_\ell, I') & \longrightarrow & A(I') \\
\downarrow & & \downarrow & & \downarrow \\
H^1(\mathbb{Q}, I) & \longrightarrow & \oplus_\ell H^1(\mathbb{Q}_\ell, I) & \longrightarrow & A(I) \\
\downarrow & & \downarrow & & \\
H^1(\mathbb{Q}, G^{\mathrm{ab}}) & \longrightarrow & \oplus_\ell H^1(\mathbb{Q}_\ell, G^{\mathrm{ab}}) & &
\end{array}$$

where the sums are over all primes ℓ of $\mathbb{Q}$.

Now return to φ and φ_ℓ. Because of our assumption (7.3.1), $Hom(\varphi(\ell), \varphi_\ell)$ is a torsor over $I_{\mathbb{Q}_\ell}$, and we write e_ℓ for its class in $H^1(\mathbb{Q}_\ell, I)$. Because of our assumption (7.3.2), each e_ℓ maps to zero in $H^1(\mathbb{Q}_\ell, G^{\mathrm{ab}})$, and so we can lift e_ℓ to an element $e'_\ell \in H^1(\mathbb{Q}_\ell, I')$. Let $e(\varphi, (\varphi_\ell))$ be the image of the family (e'_ℓ) in $\mathfrak{k}(I/\mathbb{Q})$. Clearly it is independent of the choice of the e'_ℓ.

Proposition 7.4. *There exists a φ_1 satisfying such that the conditions (7.2) if and only if $e(\varphi, (\varphi_\ell)) = 1$.*

Proof. If $e(\varphi, (\varphi_\ell)) = 1$, then we can modify the choice of the e'_ℓ's so that the family (e'_ℓ) maps to zero in $A(I')$, and so arises from an element $e_0 \in H^1(\mathbb{Q}, I')$. Take $\varphi_1 = \varphi \cdot e_0$. □

A criterion for γ_0 to arise from an admissible pair. When the Shimura variety is defined by a torus T, we saw in (5.9) that (φ_x, ε), $\varepsilon \in T(\mathbb{Q})$, is an admissible pair if and only if ε and $\mathrm{Tr}_{B(\mathbb{F}_q)/\mathbb{Q}_p}(\mu_x)$ have the same image in $X_*(T)$. We want to generalize this to other Shimura varieties.

Consider an element $\gamma_0 \in G(\mathbb{Q})$, and write I_0 for G_{γ_0}. Let S be the largest split torus in the centre of $(I_0)_{\mathbb{Q}_p}$, and let H be the centralizer of S in $G_{\mathbb{Q}_p}$; thus

$$S \subset Z(I_0)_{\mathbb{Q}_p} \subset H \subset G_{\mathbb{Q}_p}.$$

Because G is unramified over $\mathbb{Q}_p$, so also is H, and Proposition B.16 provides us with a map

$$\lambda_H \colon H(\mathbb{Q}_p) \to \pi_1(G)^{\Gamma(p)}$$

where $\Gamma(p) = \mathrm{Gal}(\mathbb{Q}_p^{\mathrm{al}}/\mathbb{Q}_p)$. On the other hand, X provides us with a conjugacy class $\mathfrak{c}(X)_{B(\mathbb{F})}$ of cocharacters of G. Assume that there exists an element $\mu \in \mathfrak{c}(X)_{B(\mathbb{F})}$ factoring through $H_{B(\mathbb{F})}$. Then μ factors through a maximal torus T of $H_{B(\mathbb{F})}$, and so defines an element of $z \in \pi_1(H)$. It is stable under $\mathrm{Gal}(B(\mathbb{F})/E_v)$, and so $\mathrm{Tr}_{E_v/\mathbb{Q}_p} z$ is fixed by $\Gamma(p)$.

Proposition 7.5. *There exists an admissible pair (φ, ε) with ε conjugate to γ_0 if and only if z exists and*

$$\lambda_H(\gamma_0) = \mathrm{Tr}_{B(\mathbb{F}_q)/\mathbb{Q}_p}\, z \text{ in } \pi_1(H)^{\Gamma(p)}.$$

Proof. See [Langlands and Rapoport 1987, 5.21, p190]. □

Definition of the Kottwitz invariant. Consider a triple $(\gamma_0; \gamma, \delta)$ satisfying the conditions (6.1). We wish to attach to $(\gamma_0; \gamma, \delta)$ an invariant $\alpha(\gamma_0; \gamma, \delta) \in \mathfrak{k}^\vee(I_0/\mathbb{Q})$.

First consider a prime $\ell \neq p, \infty$. Choose a $g \in G(\mathbb{Q}_\ell^{\mathrm{al}})$ such that $g\gamma_0 g^{-1} = \gamma_\ell$. Then

$$\tau \mapsto g^{-1} \cdot \tau g \colon \Gamma(\ell) \longrightarrow I_0(\mathbb{Q}_\ell^{\mathrm{al}})$$

is a 1-cocycle. From its construction, we see that its cohomology class lies in the kernel of

$$H^1(\mathbb{Q}_\ell, I_0) \to H^1(\mathbb{Q}_\ell, G).$$

From (B.22) we have a commutative diagram:

$$\begin{tikzcd} G^{\mathrm{ab}}(\mathbb{Q}_\ell) \arrow[r] & H^1(\mathbb{Q}_\ell, I_0') \arrow[r] \arrow[d, "\approx"] & H^1(\mathbb{Q}_\ell, I_0) \arrow[r] & H^1(\mathbb{Q}_\ell, G^{\mathrm{ab}}) \\ & A_\ell(I_0'). & & \end{tikzcd}$$

Lift the cohomology class to $H^1(\mathbb{Q}_\ell, I_0')$ and map it to $A_\ell(I_0')$; we then obtain an element $\alpha_\ell \in A_\ell(I_0')$ of whose image α_ℓ' in $\mathfrak{k}^\vee(I_0/\mathbb{Q})$ is independent of the choice of the lifting. It is zero for almost all ℓ.

Next consider $\ell = p$. We are assuming $\mathcal{N}\delta$ is conjugate to γ_0 in $G(\mathbb{Q}_p^{\mathrm{al}})$. A theorem of Steinberg shows that $H^1(B, I_0) = 0$, and therefore $\mathcal{N}\delta$ is conjugate to γ_0 in $G(B)$. Choose a $c \in G(B)$ such that $c\gamma_0 c^{-1} = \mathcal{N}\delta$. Define $b \in G(B)$ by putting $b = c^{-1}\delta\sigma c$. On applying σ to the equation $c\gamma_0 c^{-1} = \mathcal{N}\delta$, we find that $b \in I_0(B)$. Since c is well-defined up to right multiplication by an element of $I_0(B)$, the element $b \in I_0(B)$ is well-defined up to σ-conjugacy in $I_0(B)$, and hence determines a well-defined element of $\mathbb{B}(I_{0,\mathbb{Q}_p})$. Using the map

$$\mathbb{B}(I_0) \to \pi_1(I_0)_{\Gamma(p)}$$

of (B.27), we obtain an element $\alpha_p \in \pi_1(I_0)_{\Gamma(p)}$. Because of condition (6.1.4), the image of α_p in $\pi_1(G)_\Gamma$ is $[-\mu_x]$.

Finally we consider $\ell = \infty$. Choose an elliptic maximal torus T of $G_\mathbb{R}$ containing γ_0. Then T is a maximal torus in I_0 as well, and we can choose an $x \in X$ such that the image of h_x is contained in $T_\mathbb{R}$. Now μ_x defines an element of $\pi_1(I_0)_{\Gamma(\infty)}$ whose image in $\pi_1(G)_\Gamma$ is $[\mu_x]$.

Consider the diagram:

$$\begin{tikzcd} \pi_1(I_0')_{\Gamma(p)} \arrow[r] \arrow[d] & \pi_1(I_0)_{\Gamma(p)} \arrow[r] \arrow[d] & \pi_1(G)_{\Gamma(p)} \arrow[r] \arrow[d] & 0 \\ \pi_1(I_0')_\Gamma \arrow[r] & \pi_1(I_0)_\Gamma \arrow[r] & \pi_1(G)_\Gamma \arrow[r] & 0 \\ \pi_1(I_0')_{\Gamma(\infty)} \arrow[r] \arrow[u] & \pi_1(I_0)_{\Gamma(\infty)} \arrow[r] \arrow[u] & \pi_1(G)_{\Gamma(\infty)} \arrow[r] \arrow[u] & 0. \end{tikzcd}$$

The sum of the images of α_p and α_∞ in $\pi_1(I_0)_\Gamma$ maps to zero in $\pi_1(G)_\Gamma$ and so lifts to an element of $\pi_1(I_0')_\Gamma$ whose image $(\alpha_p + \alpha_\infty)'$ in $\mathfrak{k}^\vee(I_0/\mathbb{Q})$ is independent of the choice of the lifting. Define

$$\alpha(\gamma_0; \gamma, \delta) = (\alpha_p + \alpha_\infty)' + \textstyle\sum_{\ell \neq p,\infty} \alpha_\ell'.$$

Lemma 7.6. *Let* $(\gamma_0; \gamma, \delta)$ *and* $(\gamma_0'; \gamma', \delta')$ *be equivalent triples, and let* $\gamma_0' = \operatorname{ad} g(\gamma_0)$. *Then* $\operatorname{ad} g$ *defines a map* $\mathfrak{k}^\vee(I_{\gamma_0}/\mathbb{Q}) \to \mathfrak{k}^\vee(I_{\gamma_0'}/\mathbb{Q})$ *which sends* $\alpha(\gamma_0; \gamma, \delta)$ *onto* $\alpha(\gamma_0'; \gamma', \delta')$.

Proof. Omitted. □

Definition 7.7. A triple $(\gamma_0; \gamma, \delta)$ satisfying the conditions (6.1) and such that $\alpha(\gamma_0; \gamma, \delta) = 1$ is called a ***Frobenius triple***.[7]

Remark 7.8. The condition $\alpha(\gamma_0; \gamma, \delta) = 1$ implies (6.1.5), i.e., in the presence of the first condition the second may be omitted.

A characterization of effective triples.

Theorem 7.9. *A triple* $(\gamma_0; \gamma, \delta)$ *is effective if and only if it is a Frobenius triple and satisfies the condition of (7.5).*

Proof. A routine calculation shows that the condition is necessary. Conversely, consider a Frobenius triple $(\gamma_0; \gamma, \delta)$ satisfying the condition of (7.5). After replacing γ_0 by a conjugate element, we can assume that there is a special pair $(T, x) \subset (G, X)$ such that $\gamma_0 \in T(\mathbb{Q})$. Now the condition $\alpha(\gamma_0; \gamma, \delta) = 1$ and (7.4) imply that we can modify φ_x to obtain a φ_1 giving rise to $(\gamma_0; \gamma, \delta)$. □

Corollary 7.10. *If the Main Conjecture 4.4 is true for* $Sh_p(G, X)$, *then*

$$\textstyle\sum_{t'} \operatorname{Tr}(\mathcal{T}(g)^{(r)} | \mathcal{V}_t(\xi)) = \sum_{(\gamma_0; \gamma, \delta)} c(\gamma_0) \cdot I(\gamma_0; \gamma, \delta) \cdot \operatorname{Tr} \xi(\gamma_0)$$

where the second sum is over equivalence classes of Frobenius triples $(\gamma_0; \gamma, \delta)$.

Proof. Combine (6.15) with the theorem, noting that if a Frobenius triple does not satisfy the condition of (7.5) then it contributes zero to the sum on the right. □

Remark 7.11. The formula in (7.11) is exactly the formula (3.1) of [Kottwitz 1990], except that he does not assume the weight to be defined over $\mathbb{Q}$. As we noted in the introduction, probably this condition can be dropped throughout the article if the pseudomotivic groupoid is replaced by the quasimotivic groupoid.

Comparison with Kottwitz's definition. For the convenience of the reader, we verify that $\mathfrak{k}^\vee(I_0/\mathbb{Q})$, as defined above, is the dual of the group $\mathfrak{k}(I_0/\mathbb{Q})$ defined in [Kottwitz, 1986, 4.6]. For simplicity, we continue to assume that G^{der} is simply connected. We assume the reader is familiar with the theory of the dual group (see the end of Appendix B).

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[7] This notion is a modification, due to Kottwitz, of Langlands's notion of a Frobenius pair.

As before, let I_0 be the centralizer of a semisimple element γ_0 of G, and let I_0' be the centralizer of γ_0 in G^{der}, i.e., $I_0' = I_0 \cap G^{\mathrm{der}}$. From the exact sequence

$$0 \longrightarrow I_0' \longrightarrow I_0 \longrightarrow G^{\mathrm{ab}} \longrightarrow 0, \qquad (*)$$

and the observation that $Z(G^\vee) = (G^{\mathrm{ab}})^\vee$, we obtain an exact sequence

$$0 \to Z(G^\vee) \to Z(I_0^\vee) \to Z(I_0'^\vee) \to 0.$$

The boundary map

$$Z(I_0'^\vee)^\Gamma \to H^1(\mathbb{Q}, Z(G^\vee))$$

factors through $\pi_0(Z(I_0'^\vee)^\Gamma)$, and Kottwitz defines $\mathfrak{k}(I_0/\mathbb{Q})$ to be the subgroup of $\pi_0(Z(I_0'^\vee)^\Gamma)$ consisting those elements whose image in $H^1(\mathbb{Q}, Z(G^\vee))$ is locally trivial at all primes of $\mathbb{Q}$. Recall (B.5) that, for any group H, $A(H)$ is the dual of $\pi_0(Z(H^\vee)^\Gamma)$, and so we can restate the definition as follows: $\mathfrak{k}(I_0/\mathbb{Q})$ is the subgroup of $A(I_0')^\vee$ consisting of those elements whose image in $H^1(\mathbb{Q}, Z(G^\vee))$ is locally trivial. Consider the diagram

$$\begin{tikzcd}
\mathfrak{k}(I_0/\mathbb{Q}) \arrow[r] & A(I_0')^\vee \arrow[d] & \\
A_\ell(I_0)^\vee \arrow[r] & A_\ell(I_0')^\vee \arrow[r] & H^1(\mathbb{Q}_\ell, Z(G^\vee))
\end{tikzcd}$$

where we have written $A_\ell(H)$ for $A(H_{\mathbb{Q}_\ell})$. The bottom row is exact (see [Kottwitz 1984a, 2.3]), and so $\mathfrak{k}(I_0/\mathbb{Q})$ is the subgroup of $A(I_0')^\vee$ of elements whose image in $A_\ell(I_0')^\vee$ lifts to $A_\ell(I_0)^\vee$ for all ℓ. When we take duals,

$$\begin{tikzcd}
A(I_0') \arrow[r] & \mathfrak{k}(I_0/\mathbb{Q})^\vee \arrow[r] & 0 \\
A_\ell(I_0') \arrow[u] \arrow[r] & A_\ell(I_0),
\end{tikzcd}$$

we see that the dual $\mathfrak{k}(I_0/\mathbb{Q})^\vee$ of $\mathfrak{k}(I_0/\mathbb{Q})$ is the quotient of $A(I_0')$ by the subgroup of elements that are images of elements of $A_\ell(I_0)$ mapping to zero in $A_\ell(I_0)$. In other words, if we let $K(\ell) = \mathrm{Ker}(A_\ell(I_0') \to A_\ell(I_0))$, then $\mathfrak{k}^\vee(I_0/\mathbb{Q})$ is the quotient of $A(I_0)$ by the subgroup generated by the images of the groups $K(\ell)$.

Let ℓ be a prime of $\mathbb{Q}$, and consider the diagram:

$$\begin{tikzcd}
 & H^1(\mathbb{Q}_\ell, I_0'^{\mathrm{sc}}) \arrow[r] \arrow[d] & H^1(\mathbb{Q}_\ell, I_0^{\mathrm{sc}}) \arrow[d] \\
G^{\mathrm{ab}}(\mathbb{Q}_\ell) \arrow[r] & H^1(\mathbb{Q}_\ell, I_0') \arrow[r] \arrow[d, "\alpha_{I_0'}"] & H^1(\mathbb{Q}_\ell, I_0) \arrow[d, "\alpha_{I_0}"] \\
 & A_\ell(I_0') \arrow[r] \arrow[d] & A_\ell(I_0) \arrow[d] \\
 & ? \arrow[r] & ?.
\end{tikzcd}$$

When ℓ is finite, the maps α are isomorphisms (see B.22), and so

$$K(\ell) = \mathrm{Im}(G^{\mathrm{ab}}(\mathbb{Q}_\ell) \to A_\ell(I_0')).$$

Now suppose $\ell = \infty$, so that $\mathbb{Q}_\ell = \mathbb{R}$. It is clear from the sequence $(*)$ that $I_0'^{\mathrm{der}} = I_0^{\mathrm{der}}$, and so the top horizontal arrow is an isomorphism. The bottom horizontal arrow is

$$\pi_1(I_0') \to \pi_1(I_0),$$

which (B.2a) shows to be injective, and a diagram chase shows that

$$K(\infty) = \mathrm{Im}(G^{\mathrm{ad}}(\mathbb{R}) \to A_\infty(I_0')).$$

Thus we can conclude that $\mathfrak{k}(I_0/\mathbb{Q})^\vee$ is the quotient of $A(I_0')$ by the subgroup generated by the images of the groups $G(\mathbb{Q}_\ell)$, i.e., that $\mathfrak{k}(I_0/\mathbb{Q})^\vee = \mathfrak{k}^\vee(I_0/\mathbb{Q})$.

We leave it as an exercise to the reader to prove that the our definition of $\alpha(\gamma_0;\gamma,\delta)$ agrees with that of [Kottwitz 1990, §2]; in fact, apart from the description of $\mathfrak{k}(I_0/\mathbb{Q})$, our definition is identical to that of Kottwitz.

Notes. The sources have been noted in the text.

8. STABILIZATION

In this section, we summarize the results of [Kottwitz 1990, §4–§7] concerning the stabilization of the expression on the right in (7.10):

$$\sum_{(\gamma_0;\gamma,\delta)} c(\gamma_0)\cdot \mathrm{vol}(I(\mathbb{Q})\backslash I(\mathbb{A}_f)\cdot O_\gamma(f^p)\cdot TO_\delta(\phi_r)\cdot \mathrm{Tr}\,\xi(\gamma_0) \tag{8.0.1}$$

Here the sum is over a set of representatives for the equivalence classes of Frobenius triples $(\gamma_0;\gamma,\delta)$. As does Kottwitz, we assume that G^{der} is simply connected and that the largest $\mathbb{R}$-split subtorus of $Z(G)$ is already split over $\mathbb{Q}$.

Statement of the results. Write $\xi_{\mathbb{C}}$ for the representation $\xi \otimes_{\mathbb{Q}} \mathbb{C}$. Fix an embedding $\mathbb{Q}_\ell \hookrightarrow \mathbb{C}$ and write $f^p_{\mathbb{C}}$ for the composite of f^p with $\mathbb{Q}_\ell \hookrightarrow \mathbb{C}$. There is a unique Haar measure di_∞ such that $di^p \cdot di_p \cdot di_\infty$ is the canonical measure on $I(\mathbb{A})$. Let A_G be the maximal $\mathbb{Q}$-split subtorus of the centre of G, and let $A_G(\mathbb{R})^+$ denote the identity component of the topological group $A_G(\mathbb{R})$. Write e(I(v)) for the signs defined in [Kottwitz 1983], and set

$$e(\gamma,\delta)=\prod_v e(I(v)).$$

If $\alpha(\gamma_0;\gamma,\delta)=1$, so that there exists a group I over $\mathbb{Q}$ whose localizations are the groups I_ℓ, then $e(\gamma,\delta)=1$ by the main result (ibid.)

As in [Kottwitz 1990, §4], (8.0.1) can be rewritten as

$$\tau(G)\sum_{\gamma_0}\sum_{\kappa}\sum_{(\gamma,\delta)} <\alpha(\gamma_0;\gamma,\delta),\kappa> \cdot e(\gamma,\delta)\cdot O_\gamma(f^p_{\mathbb{C}})\cdot TO_\delta(\phi_r)\cdot$$
$$\mathrm{Tr}\xi_{\mathbb{C}}(\gamma_0)\cdot \mathrm{vol}(A_G(\mathbb{R})^+\backslash I(\infty)(\mathbb{R}))^{-1},$$

where the first sum is over a set of representatives for the $G(\mathbb{Q}^{\mathrm{al}})$-conjugacy classes of semisimple elements $\gamma_0 \in G(\mathbb{Q})$ that are elliptic in $G_{\mathbb{R}}$, the second sum is over the elements of κ of $\mathfrak{k}(I_0/\mathbb{Q})$, and the third sum is over a set of representatives for the equivalence classes of pairs (γ,δ) such that $(\gamma_0;\gamma,\delta)$ is a Frobenius triple. Here $\tau(G)$ is the Tamagawa number of G.

We assume that the reader (unlike the author) is familiar with elliptic endoscopic triples (H,s,η_0)—see Kottwitz (1984a), §7.

Assuming three standard conjectures on the transfer of functions on p-adic groups, and a global hypothesis, which we list below, Kottwitz constructs the following functions.

(a) A function $h^p \in C^\infty_p(G(\mathbb{A}^p_f))$ such that

$$SO_{\gamma_H}(h^p)=\sum_\gamma \Delta^p(\gamma_H,\gamma)\cdot e^p(\gamma)\cdot O_\gamma(f^p_{\mathbb{C}})$$

for every (G,H)-regular semisimple element $\gamma_H \in H(\mathbb{A}^p_f)$. Here $e^p(\gamma)$ is a product of the signs $e(G_{\gamma(\ell)})$ for $\ell \neq p,\infty$, and $\Delta^p(\gamma_H,\gamma)$ is a product of local transfer factors $\Delta_\ell(\gamma_H,\gamma)$ for $\ell \neq p,\infty$.

(b) A function $h_p \in C^\infty_c(H(\mathbb{Q}_p))$ such that

$$SO_{\gamma_H}(h_p)=\sum_\delta <\beta(\gamma_0;\delta),s> \cdot \Delta_p(\gamma_H,\gamma)\cdot e(I)\cdot TO_\delta(\phi_r).$$

for every (G, H)-regular semisimple character $\gamma_H \in H(\mathbb{Q}_p)$. If η is unramified, then h_p belongs to the Hecke algebra $\mathcal{H}(H(\mathbb{Q}_p), K_H)$; in general, it is a quasi-character on $H(\mathbb{Q}_p)$ times a function in $\mathcal{H}(H(\mathbb{Q}_p), K_H)$.

(c) A function $h_\infty \in C^\infty(H(\mathbb{R}))$, compactly supported modulo $A_G(\mathbb{R})^+$ such that

$$SO_{\gamma_H}(h_\infty)$$
$$=< \beta(\gamma_0 : \delta), s > \cdot \Delta_\infty(\gamma_H, \gamma_0) \cdot e(I) \cdot \mathrm{Tr}\ \xi_{\mathbb{C}}(\gamma_0) \cdot \mathrm{vol}(A_G(\mathbb{R})^+ \backslash I(\mathbb{R}))^{-1}$$

for every (G,H)-regular semisimple $\gamma_H \in H(\mathbb{R})$ which is elliptic, and is zero for nonelliptic such elements. Here γ_0 is an element of $T(\mathbb{R})$ that comes from γ_H and I is a certain inner form of I_{γ_0}.

Write $\iota(G, H)$ for the positive rational number

$$\iota(G, H) = \tau(G) \cdot \tau(H)^{-1} \cdot \mathrm{Card}(\mathrm{Aut}(H, s, \eta)/H^{\mathrm{ad}}(\mathbb{Q}))^{-1},$$

and $ST_e^*(h)$ for the (G, H)-regular $\mathbb{Q}$-elliptic part of the stable trace formula for (H, h),

$$ST_e^*(h) = \sum_{\gamma_H} \mathrm{Card}((H_{\gamma_H}/H_{\gamma_H}^0)(\mathbb{Q}))^{-1} \cdot \tau(H) \cdot SO_{\gamma_H}(h).$$

Theorem 8.1. *Assuming the existence of functions h as above, (8.0.1) is equal to*

$$\sum_{\mathcal{E}} \iota(G, H) \cdot ST_e^*(h) \tag{8.0.3}$$

where the sum is over a set $\mathcal{E}$ of representatives for the isomorphism classes of the elliptic endoscopic triples for G.

Proof. This is Kottwitz (1990), 7.2. □

Comparison with Rogawski's article. It is left as an exercise to the Editors to reconcile (8.0.3) with the notation in Rogawski's article.

The conjectures used in the proof of 8.1. In constructing h^p Kottwitz assumes the "fundamental lemma" for H, G, and the unit element of the unramified Hecke algebra of G at all but a finite number of primes of $\mathbb{Q}$, and he assumes the following conjecture.

Conjecture 8.2 [Kottwitz 1986, 5.5]. *Let F be a local field of characteristic zero, and let G be a connected reductive group over F. Let (H, s, η) be an endoscopic triple for G, and choose an extension of $\eta : H^\vee \to G^\vee$ to an L-homomorphism*

$\eta' : {}^LH \to {}^LG$. There should be a correspondence (f, f^H) between functions $f \in C_c^\infty(G(F))$, $f^H \in C_c^\infty(H(F))$, such that

$$SO_{\gamma_H}(f^H) = \sum_\gamma \Lambda(\gamma_H, \gamma) \cdot O_\gamma(f)$$

for every G-regular semisimple element $\gamma_H \in H(F)$. The sum runs over a set (possibly empty) of representatives for the conjugacy classes in $G(F)$ belonging to the $G(F^{\mathrm{al}})$-conjugacy class obtained from γ_H in the following way: choose a maximal torus T_H in $H_{F^{\mathrm{al}}}$ containing γ_H and an embedding $j : T_H \hookrightarrow G_{F^{\mathrm{al}}}$ (canonical up to $G(F^{\mathrm{al}})$- conjugacy); the conjugacy class is that containing $j(\gamma_H)$. The transfer factors $\Lambda(\gamma_H, \gamma)$ (complex numbers) and the correspondence will depend on the choice of η'.

In constructing h_p, Kottwitz assumes the "fundamental lemma" for the homomorphism of Hecke algebras

$$\mathcal{H}(G(F), K_F) \to \mathcal{H}(H(\mathbb{Q}_p), K_H)$$

[Kottwitz 1990, p180].

Finally, he assumes the "global hypothesis" for the transfer factors.

There is a statement of the "fundamental lemma" (in the case that G^{der} is simply connected) in the introduction to Kottwitz (1986). The[8] lemma is proved for the groups of interest to the seminar in Blasius and Rogawski (1991) (using some calculations from Kottwitz (1990)).

Conjecture 8.2 was proved in the archimedean case by Shelstad for G-regular pairs. It is only known in a few p-adic cases. For p-adic unitary groups in three variables, it is proved for G-regular pairs in Langlands and Shelstad (1989). The archimedean and p-adic cases are quoted in Rogawski (1990), 4.9.1, in the G-regular case, and the extension to the (G, H)-regular case is carried out in Chapter 8 of the same work.

Concerning the global hypothesis, Langlands and Shelstad define local transfer factors for pairs of G-regular elements, well-defined up to a non-zero scalar, and they show that there is a choice such that the global hypothesis is satisfied. Kottwitz needs transfer factors for pairs of (G, H)-regular elements, and he needs to know that they can be chosen so that the global hypothesis holds. Thus it needs to be checked that the local transfer factors of Langlands and Shelstad extend by continuity to (G, H)-regular pairs, and that the global factors satisfying the global

[8] I am grateful to Jon Rogawski for a message on which the rest of this section is based.

hypothesis for G-regular pairs also satisfy it for (G,H)-regular pairs. (According to Rogawski) this will probably be straightforward, but has not been written down. In any case, it is easy to check everything for unitary groups in three variables, because of the explicit form of the transfer factors.

Notes. The sources have been noted in the text.

APPENDIX A. GROUPOIDS AND TENSOR CATEGORIES

The notion of a groupoid is a natural generalization of that of a group. Affine groupoids classify nonneutral Tannakian categories in exactly the same way that affine group schemes classify neutral Tannakian categories.

Throughout this section, S_0 is the spectrum of a field k_0 of characteristic zero, and S is an affine scheme over S_0.

Groupoids in sets. A ***groupoid in sets*** is a category in which every morphism has an inverse. Thus to give a groupoid in sets is to give a set S (of objects), a set G (of arrows), two maps $t, s\colon G \rightrightarrows S$ (sending an arrow to its target and source respectively), and a law of composition (map over $S \times S$)

$$\circ\colon G \underset{s,S,t}{\times} G \to G \text{ where } G \underset{s,S,t}{\times} G = \{\, (h,g) \in G \times G \mid s(h) = t(g) \,\}$$

satisfying the following conditions: each object has an identity morphism; composition of arrows is associative; each arrow has an inverse. We often refer to G as a ***groupoid acting on*** S.

A groupoid is said to be ***transitive*** if the map

$$(t,s)\colon G \to S \times S,$$

is surjective, i.e., if for every pair of objects (b,a) of S there exists an arrow $a \to b$.

Example A.1. A group G defines a groupoid in sets as follows: take S to be any one-element set, so that there are unique maps $t, s\colon G \to S$, and take $\circ$ to be multiplication on G. Conversely a groupoid G acting on a one-point set S is a group.

Let G be a transitive groupoid. We often regard G as a set over $S \times S$ using the map (t,s). Write $G_{b,a}$ for the fibre of G over (b,a); thus

$$G_{b,a} = \{g \in G \mid s(g) = a,\ t(g) = b\} = \{g \mid g\colon a \to b\} = \mathrm{Hom}(a,b),$$

and there is a law of composition

$$G_{c,b} \times G_{b,a} \to G_{c,a}.$$

This law makes $G_a =_{df} G_{a,a}$ into a group and $G_{b,a}$ into a right principal homogeneous space for G_a. The choice of an element $u_{b,a} \in G_{b,a}$ defines an isomorphism $\operatorname{ad} u_{b,a}\colon G_a \to G_b$.

The ***kernel*** G^Δ of G is the family $(G_a)_{a\in S}$. It can be thought of as a relative group over S.

If G is transitive and G_a is commutative for one (hence all) $a \in S$, then we say that G is ***commutative.*** In this case the isomorphism $\operatorname{ad} u_{b,a}\colon G_a \to G_b$ is independent of the choice of $u_{b,a}$, and so there is a canonical isomorphism $G_0 \times S \to G^\Delta$ for any $0 \in S$, i.e., G^Δ is a constant group over S.

Example A.2. Let S be a topological space. The ***fundamental groupoid*** Π of S is the groupoid acting on S for which $\Pi_{b,a}$ is the set of paths from a to b taken up to homotopy. The law of composition is the usual composition of paths. In this case, the group Π_a is the fundamental group $\pi_1(S, a)$.

A morphism $\varphi\colon P \to G$ of groupoids acting on S is a function that, together with the identity map $S \to S$, is a functor of categories. Let P and G be two groupoids acting transitively on S, and let φ and ψ be morphisms $P \to G$. A ***morphism*** $\alpha\colon \varphi \to \psi$ is a morphism of functors. Thus it is a family of arrows $\alpha_a\colon a \to a$ in G, indexed by the elements of S, such that the diagrams

$$\begin{tikzcd} a \arrow[r, "\alpha_a"] \arrow[d, "\varphi(p_{b,a})"] & a \arrow[d, "\psi(p_{b,a})"] \\ b \arrow[r, "\alpha_b"] & b \end{tikzcd}$$

commute for all $p_{b,a} \in P_{b,a}$. Note that every morphism $\alpha\colon \varphi \to \psi$ is an isomorphism. Write $\operatorname{Isom}(\varphi, \psi)$ for the set of (iso)morphisms $\varphi \to \psi$. An element $(\alpha_a)_{a\in S}$ of $\operatorname{Isom}(\varphi, \psi)$ is determined by a single component α_a, because for any $p_{b,a} \in P_{b,a}$, $\alpha_b = \psi(p_{b,a})\circ\alpha_a\circ\varphi(p_{b,a})^{-1}$. Let $\operatorname{Aut}(\varphi) = \operatorname{Isom}(\varphi, \varphi)$. It is a group, and $\operatorname{Isom}(\varphi, \psi)$ is either empty or is a right principal homogeneous space for $\operatorname{Aut}(\varphi)$ over S.

Assume that P is commutative, so that $P^\Delta = P_0 \times S$, and let $I(\varphi)$ be the subset of G such that

$$I(\varphi)_{b,a} = \{g\colon a \to b \mid g \circ \varphi(p) = \varphi(p) \circ g,\ \text{all } p \in P_0\}.$$

The restrictions of t, s, and $\circ$ to $I(\varphi)$ define on it the structure of a groupoid acting on S. The group $I(\varphi)_a$ is the centralizer of $\varphi(P_a)$ in G_a, and an element $u_{b,a} \in P_{b,a}$ defines an isomorphism $\operatorname{ad} \varphi(u_{b,a})$ from $I(\varphi)_a$ to $I(\varphi)_b$ that is independent of $u_{b,a}$; thus $I(\varphi)^\Delta = I(\varphi)_0 \times S$ for any $0 \in S$. Note that $\operatorname{Aut}(\varphi) = I(\varphi)^\Delta$.

Groupoids in schemes. Recall that $S_0 = \operatorname{Spec} k_0$. For any affine scheme S over S_0, an ***S_0-groupoid (in schemes) acting on S*** is a scheme $\mathfrak{G}$ over S_0 together with two S_0-morphisms $t, s\colon \mathfrak{G} \rightrightarrows S$ and a law of composition (morphism of $S \times_{S_0} S$-schemes)

$$\circ\colon \mathfrak{G} \underset{s,S,t}{\times} \mathfrak{G} \to \mathfrak{G}$$

such that, for all schemes T over S_0, $(S(T), \mathfrak{G}(T), (t,s), \circ)$ is a groupoid in sets. We also refer to $\mathfrak{G}$ as a ***k_0-groupoid acting on S***, or as an ***S/S_0-groupoid.***

A groupoid is said to be ***affine*** if it is an affine scheme, and it is ***algebraic*** if it is of finite type over $S \times_{S_0} S$. An affine groupoid is the projective limit of its algebraic quotients.

Henceforth, all groupoids will be affine.

A groupoid is said to act ***transitively*** on S, or be a ***transitive*** S/S_0-groupoid, if the map $(t,s)\colon \mathfrak{G} \to S \times_{S_0} S$ makes it into a faithfully flat $S \times_{S_0} S$-scheme.

Example A.3. An S_0/S_0-groupoid is just an affine group scheme over S_0. It is automatically transitive.

For a scheme $(b,a)\colon T \to S \times_{S_0} S$ over $S \times_{S_0} S$, we write $\mathfrak{G}_{b,a}$ for $(b,a)^*\mathfrak{G}$. Note that b and a are objects of the category $S(T)$, and $\mathfrak{G}_{b,a}$ can be thought of as the scheme of arrows $a \to b$,

$$\mathfrak{G}_{b,a} = \text{“}\mathrm{Hom}(a,b)\text{”}.$$

The law of composition provides morphisms (of schemes over T)

$$\mathfrak{G}_{c,b} \times_T \mathfrak{G}_{b,a} \to \mathfrak{G}_{c,a}.$$

This law makes $\mathfrak{G}_a =_{df} \mathfrak{G}_{a,a} =_{df} (a,a)^*\mathfrak{G}$ into an affine group scheme over T, which is flat if $\mathfrak{G}$ is transitive.

A ***morphism*** $\alpha\colon \mathfrak{G} \to \mathfrak{G}'$ of S/S_0-groupoids is a morphism of $S \times_{S_0} S$-schemes such that, for all S_0-schemes T, $\alpha(T)$ is a morphism of groupoids in sets acting on $S(T)$.

Example A.4. Let V be a locally free $\mathcal{O}_S$-module of finite rank. For a scheme $(b,a)\colon T \to S \times_{S_0} S$ over $S \times_{S_0} S$, let $\mathit{Isom}(a^*V, b^*V)$ be the scheme representing the functor that sends a T-scheme $c\colon T' \to T$ to $\mathrm{Isom}_{\mathcal{O}_{T'}}((ac)^*V, (bc)^*V)$. There is an affine groupoid $\mathfrak{G}(V)$ such that, for any scheme $(b,a)\colon T \to S \times_{S_0} S$ over $S \times_{S_0} S$,

$$\mathfrak{G}(V)_{b,a} = \mathit{Isom}(a^*V, b^*V).$$

It is transitive.

More generally, suppose V has a tensor structure, i.e., a family $\mathfrak{t} = (t_i)$ with $t_i \in \Gamma(S, V^{\otimes r_i} \otimes V^{\vee \otimes s_i})$ for some r_i and s_i. Then we can define an affine groupoid $\mathfrak{G}(V, \mathfrak{t})$ such that $\mathfrak{G}(V, \mathfrak{t})_{b,a}$ is the subscheme of $Isom(a^*V, b^*V)$ whose points are the isomorphisms preserving the tensors (or preserving the tensors up to a constant). It need not be transitive.

Pull-backs of groupoids. Let $\mathfrak{G}$ be an S/S_0-groupoid, and consider a commutative diagram:

$$\begin{array}{ccc} S & \xleftarrow{u} & S' \\ \downarrow & & \downarrow \\ S_0 & \longleftarrow & S_0'. \end{array}$$

If $\mathfrak{G}$ is an S/S_0-groupoid, then the pull-back of $\mathfrak{G}$ relative to

$$S \times_{S_0} S \xleftarrow{u \times u} S' \times_{S_0'} S'$$

is an S'/S_0'-groupoid.

Example A.5.

(a) Let G_0 be an affine group scheme over S_0, regarded as an S_0/S_0-groupoid. On pulling it back to S, we obtain an S/S_0-groupoid

$$\mathfrak{G}_{G_0} =_{df} G_0 \times_{S_0} (S \times_{S_0} S),$$

which is called the ***neutral groupoid*** defined by G_0. In the special case that G_0 is the trivial group, $\mathfrak{G}_{G_0} = S \times_{S_0} S$ and is called the ***trivial*** S/S_0-groupoid.

(b) If $\mathfrak{G}$ is an S/S_0-groupoid and $u \colon T \to S$ is a morphism of affine schemes over S_0, then the inverse image of $\mathfrak{G}$ by $u \times u$ is a T/S_0-groupoid, which we denote by $\mathfrak{G}_T$.

(c) If $\mathfrak{G}$ is an S/S_0-groupoid, then the pull-back of $\mathfrak{G}$ relative to

$$S = S \times_S S \to S \times_{S_0} S$$

is an S/S-groupoid, i.e., a group scheme over S. This is the kernel of $\mathfrak{G}$ (see below).

Descent data. Let $S \to S_0$ be a morphism of schemes, and let $X \to S$ be a scheme over S. A ***descent datum*** on X relative to S/S_0 is a map

$$u\colon \mathrm{pr}_1^* X \to \mathrm{pr}_2^* X \text{ (over } S \times_{S_0} S)$$

satisfying the cocycle condition:

$$\mathrm{pr}_{13}^*(u) = \mathrm{pr}_{23}^*(u) \circ \mathrm{pr}_{12}^*(u)\colon \mathrm{pr}_1^* X \to \mathrm{pr}_3^* X \text{ (over } S \times_{S_0} S \times_{S_0} S).$$

Here pr_i is the projection onto the i^{th} factor and pr_{ij} is the projection onto the $(i,j)^{\text{th}}$ factor.

It is often easier to think of descent data in terms of points. For each S_0-scheme T and point $(a,b) \in (S \times_{S_0} S)(T)$ a descent datum gives a morphism $u_{b,a}\colon a^*X \to b^*X$ over T, and the cocycle condition asserts that for every point $(a,b,c) \in (S \times_{S_0} S \times_{S_0} S)(T)$, $u_{c,a} = u_{c,b} \circ u_{b,a}$.

The set of pairs consisting of an affine S-scheme X and an S/S_0-descent datum can be made into a category $\mathbf{Desc}(S/S_0)$ in an obvious way. An affine scheme X_0 over S_0 defines an object (X,u) of $\mathbf{Desc}(S/S_0)$ with $X = X_0 \times_{S_0} S$, and under our assumption that S_0 is the spectrum of a field and S is affine, the map $X_0 \mapsto (X,u)$ defines an equivalence of categories:

$$\mathbf{Aff}_{S_0} \to \mathbf{Desc}(S/S_0).$$

Here $\mathbf{Aff}_{S_0}$ denotes the category of affine schemes over S_0.

Let G be a group scheme over S. An isomorphism

$$u\colon \mathrm{pr}_1^* X \to \mathrm{pr}_2^* X \text{ (over } S \times_{S_0} S)$$

such that $\mathrm{pr}_{13}^*(u)$ differs from $\mathrm{pr}_{23}^*(u) \circ \mathrm{pr}_{12}^*(u)$ by an inner automorphism of G defines the structure of a S_0-***band*** on G; the structure defined by an isomorphism u' differing from u by an inner automorphism is not distinguished from that defined by u (see Deligne and [Milne 1982, p223], or [Giraud 1971, IV.1], for a more precise definition).

Kernels of groupoids. Let $\mathfrak{G}$ be an S/S_0-groupoid. The ***kernel*** of $\mathfrak{G}$ is

$$G =_{df} \mathfrak{G}^{\Delta} =_{df} \Delta^*\mathfrak{G}, \quad \Delta\colon S \to S \times_{S_0} S \text{ (diagonal morphism)}.$$

It is an affine group scheme over S, and it is faithfully flat over S if $\mathfrak{G}$ is transitive. We say that $\mathfrak{G}$ is ***commutative*** if it is transitive and G is commutative.

Let $\mathfrak{G}$ be an S/S_0-groupoid with kernel G. Then $\mathrm{pr}_2^* G$ acts on $\mathfrak{G}$ over $S \times_{S_0} S$ and makes it into a right torsor. The groupoid $\mathfrak{G}$ acts on G by conjugation:

$$(g, x) \mapsto g \circ x \circ g^{-1}, \quad g \in \mathfrak{G}(T), \quad x \in G(T).$$

Let $\mathfrak{G}$ be a transitive S/S_0-groupoid, and let u be a section of $\mathfrak{G}$ over $S \times_{S_0} S$. Then u defines a morphism

$$\varphi =_{df} \operatorname{ad} u\colon\ \mathrm{pr}_1^* \mathfrak{G}^\Delta \to \mathrm{pr}_2^* \mathfrak{G}^\Delta$$

such that $\mathrm{pr}_{13}^*(\varphi)$ differs from $\mathrm{pr}_{23}^*(\varphi) \circ \mathrm{pr}_{12}^*(\varphi)$ by an inner automorphism. If u is replaced by a different section, then φ is replaced by its composite with an inner automorphism (because $\mathfrak{G}$ is a $\mathrm{pr}_2^* \mathfrak{G}^\Delta$-torsor). Thus u defines the structure of an S_0-band on $\mathfrak{G}^\Delta$. In terms of points, $\varphi_{b,a} = \operatorname{ad}(u_{b,a})\colon \mathfrak{G}_a \to \mathfrak{G}_b$, $\mathrm{pr}_{13}^*(\varphi) = \operatorname{ad}(u_{c,a})$, and $\mathrm{pr}_{23}^*(\varphi) \circ \mathrm{pr}_{12}^*(\varphi) = \operatorname{ad}(u_{c,b} \circ u_{b,a})$.

A section u of $\mathfrak{G}$ over $S \times_{S_0} S$ will be called ***special*** if $\varphi = \operatorname{ad} u$ satisfies the cocycle condition $\mathrm{pr}_{13}^*(\varphi) = \mathrm{pr}_{23}^*(\varphi) \circ \mathrm{pr}_{12}^*(\varphi)$. Such a u defines a model G_0 of $G = \mathfrak{G}^\Delta$ over k_0. Note that if G is commutative, then every section of $\mathfrak{G}$ over $S \times_{S_0} S$ is special, and that the model G_0 of G defined by a section is independent of the choice of the section.

Let $\mathfrak{G}$ and $\mathfrak{G}'$ be groupoids acting on S. The restriction to the diagonal of a morphism $\alpha\colon \mathfrak{G} \to \mathfrak{G}'$ is a homomorphism of group schemes $\alpha^\Delta\colon \mathfrak{G}^\Delta \to \mathfrak{G}'^\Delta$. If $\mathfrak{G}$ and $\mathfrak{G}'$ have special sections u and u', and α maps u to u', then α^Δ is defined over k_0.

Let $\mathfrak{G}$ and $\mathfrak{H}$ be S/S_0-groupoids with kernels G and H, and let $\varphi\colon G \to H$ be a homomorphism. If there is given an action of $\mathfrak{G}$ on H compatible with its action on G, then the $\mathrm{pr}_2^* H$-torsor deduced from $\mathfrak{G}$ by pushing out by the morphism $pr^*\varphi : \mathrm{pr}_2^* G \to \mathrm{pr}_2^* H$ is endowed with the structure of a groupoid whose kernel is H. We denote it by $\varphi_* \mathfrak{G}$. (See [Deligne 1989, 10.8.])

Example A.6.

(a) The kernel of the neutral gerb $\mathfrak{G}_{G_0}$ is $G =_{df} G_0 \times_{S_0} S$. The identity section of G_0 over S_0 defines a canonical section of $\mathfrak{G}_{G_0}$ over $S \times_{S_0} S$. This section is special, and defines the model G_0 of G.

(b) The kernel of $\mathfrak{G}(V)$ is $\mathrm{GL}(V)$ with its canonical structure of a band, namely, the isomorphism $\operatorname{ad} u\colon\ \mathrm{pr}_1^* \mathrm{GL}(V) \to \mathrm{pr}_2^* \mathrm{GL}(V)$ defined by an isomorphism $u\colon\ \mathrm{pr}_1^* V \to \mathrm{pr}_2^* V$.

Tensor categories. A ***tensor category*** is a category $\mathbf{T}$ together with a functor $\otimes\colon \mathbf{T}\times\mathbf{T}\to\mathbf{T}$ and sufficient constraints so that the tensor product of any finite (unordered) set of objects of $\mathbf{T}$ is well defined up to a unique isomorphism. In particular, there is an ***identity object*** 1 (tensor product of the empty set of objects) with the property that

$$1\otimes X = X = X\otimes 1$$

for all objects X in $\mathbf{T}$, an ***associativity constraint*** (functorial in X, Y, Z)

$$\phi_{X,Y,Z}\colon X\otimes(Y\otimes Z)\xrightarrow{\approx}(X\otimes Y)\otimes Z,$$

and a ***commutativity constraint*** (functorial in X, Y)

$$\psi_{X,Y}\colon X\otimes Y\xrightarrow{\approx} Y\otimes X.$$

Let $(\mathbf{T},\otimes)$ and $(\mathbf{T}',\otimes')$ be tensor categories. A ***tensor functor*** from $(\mathbf{T},\otimes)$ to $(\mathbf{T}',\otimes')$ is a functor $F\colon\mathbf{T}\to\mathbf{T}'$ together with a natural isomorphism

$$c_{X,Y}\colon F(X)\otimes' F(Y)\to F(X\otimes Y)$$

compatible with constraints. In particular, for any finite family (X_i) of objects of $\mathbf{T}$, there is a well-defined isomorphism

$$c\colon\ \otimes_i' F(X_i)\to F(\otimes_i X_i).$$

A ***morphism of tensor functors*** $\gamma\colon (F,c)\to(F',c')$ is a morphism of functors commuting with tensor products, i.e., such that the diagrams

$$\begin{array}{ccc} 1 & \xrightarrow{c_1} & F(1) \\ \| & & \downarrow{\scriptstyle\gamma(1)} \\ 1 & \xrightarrow{c_1} & F'(1) \end{array} \qquad \begin{array}{ccc} F(X\otimes Y) & \xrightarrow{c_{X,Y}} & F(X)\otimes F(Y) \\ \downarrow{\scriptstyle\gamma(X\otimes Y)} & & \downarrow{\scriptstyle\gamma(X)\otimes\gamma(Y)} \\ F'(X\otimes Y) & \xrightarrow{c'_{X,Y}} & F'(X)\otimes F'(Y) \end{array}$$

commute.

Tannakian categories. Let k_0 be a field. A tensor category $(\mathbf{T},\otimes)$ together with an isomorphism $k_0\to\mathrm{End}(1)$ is said to be ***pseudo-Tannakian*** over k_0 if

A.7.1. $\mathbf{T}$ is abelian, and

A.7.2. for each X in $\mathbf{T}$, there exists an object $X^\vee$ and morphisms $\mathrm{ev}\colon X\otimes X^\vee\to 1$ and $\delta\colon 1\to X^\vee\otimes X$ such that

$$(\mathrm{ev}\otimes X)\circ(X\otimes\delta)=\mathrm{id}_X,\quad (X^\vee\otimes\mathrm{ev})\circ(\delta\otimes X^\vee)=\mathrm{id}_{X^\vee}.$$

These conditions imply that $(\mathbf{T}, \otimes)$ has an internal Hom and that $\otimes$ is k_0-bilinear and exact in each variable (Deligne 1990, 2.1–2.5): in the terminology of [Saavedra 1972], $(\mathbf{T}, \otimes)$ is a k_0-linear, rigid, abelian tensor category ACU such that $k_0 = \mathrm{End}(1)$; in the terminology of [Deligne and Milne 1982], it is a k_0-linear, rigid, abelian tensor category such that $k_0 = \mathrm{End}(1)$.

Let $(\mathbf{T}, \otimes)$ be a pseudo-Tannakian category over k_0, and let A be a k_0-algebra. A ***fibre functor*** of $\mathbf{T}$ over A is an exact faithful k_0-linear tensor functor from $\mathbf{T}$ to the category of finitely generated A-modules. A pseudo-Tannakian category over k_0 is said to be ***Tannakian*** if it possesses a fibre functor over some nonzero k_0-algebra; when it possesses a fibre functor over k_0 itself, it is said to be a ***neutral Tannakian category***.

The ***dimension*** of an object X of a pseudo-Tannakian category is the element $\mathrm{ev} \circ \delta$ of k_0. A theorem of Deligne shows that, when k_0 has characteristic zero, a pseudo-Tannakian category over k_0 is Tannakian if (and only if) the dimensions of its objects are nonnegative integers (Deligne 1990, 7.1).

The classification of Tannakian categories in terms of groupoids. A ***representation*** of a k/k_0-groupoid $\mathfrak{G}$ is a homomorphism $\varphi\colon \mathfrak{G} \to \mathfrak{G}(V)$ for some finite-dimensional vector space V over k. The category $\mathbf{Rep}(S\colon \mathfrak{G})$ of representations of $\mathfrak{G}$ has a natural tensor structure relative to which it forms a Tannakian category, and the forgetful functor is a fibre functor over k.

Let $(\mathbf{T}, \otimes)$ be a Tannakian category, and let ω be a fibre functor of $\mathbf{T}$ over an affine scheme S. Write $Aut^{\otimes}_{k_0}(\omega)$ for the functor sending an $S \times_{S_0} S$-scheme $(b, a)\colon T \to S \times_{S_0} S$ to the set of isomorphisms of tensor functors $a^*\omega \to b^*\omega$. Note that, $Aut^{\otimes}_{k_0}(\omega)^{\Delta}$, the restriction of $Aut^{\otimes}_{k_0}(\omega)$ to the diagonal, is the functor $Aut^{\otimes}_{S}(\omega)$ sending an S-scheme $a\colon T \to S$ to the set of automorphisms of the tensor functor $a^*\omega$.

Theorem A.8. *Let $(\mathbf{T}, \otimes)$ be a Tannakian category over a field k_0, and let ω be a fibre functor of $\mathbf{T}$ over a nonempty affine k_0-scheme S.*

(i) *The functor $Aut^{\otimes}_{k_0}(\omega)$ is represented by an S/S_0-groupoid $\mathfrak{G}$ which acts transitively on S.*

(ii) *The fibre functor ω defines an equivalence of tensor categories $\mathbf{T} \to \mathbf{Rep}(S\colon \mathfrak{G})$.*

Conversely, let $\mathfrak{G}$ be a k_0-groupoid acting transitively on a nonempty affine scheme S, and let ω be the forgetful fibre functor of $\mathbf{Rep}(S\colon \mathfrak{G})$; then the natural map $\mathfrak{G} \to Aut^{\otimes}_{k_0}(\omega)$ is an isomorphism.

The proof of this theorem occupies most of [Deligne 1990]. The key point in the proof of (ii) is the following theorem of Barr-Beck in category theory: let $\mathbf{A}$ and $\mathbf{B}$ be abelian categories, and let $T\colon \mathbf{A} \to \mathbf{B}$ be an exact faithful functor having a right adjoint U; then the functor T defines an equivalence of $\mathbf{A}$ with the category of pairs $(B, \cdot)$ where B is an object of $\mathbf{B}$ and $\cdot$ is a "coaction" of the "comonad" TU on B (see Deligne, 1990, §4 for a detailed statement).

Corollary A.9. *Any two fibre functors of* $\mathbf{T}$ *over* S *become isomorphic over some faithfully flat covering of* S.

Proof. Let ω_1 and ω_2 be fibre functors of $\mathbf{T}$ over S_1 and S_2 respectively. There exists a fibre functor ω over $T =_{df} S_1 \amalg S_2$ whose restriction to S_i is ω_i, $i = 1, 2$. According to (i) of the theorem, the scheme $Aut^{\otimes}_{k_0}(\omega)$ is faithfully flat over $T \times_{S_0} T$. We want to apply this statement in the case $S_1 = S_2 = S$. In general

$$T \times_{S_0} T = \amalg_{1 \leq i,j \leq 2} S_i \times_{S_0} S_j,$$

and in our case the restriction of $Aut^{\otimes}_{k_0}(\omega)$ to the subscheme

$$S \xrightarrow{\Delta} S \times_{S_0} S = S_2 \times_{S_0} S_1 \subset T \times_{S_0} T$$

is $Isom^{\otimes}_S(\omega_1, \omega_2)$, which is therefore faithfully flat over S. Consequently it acquires a section over some S' faithfully flat over S, for example, over $S' = Isom^{\otimes}_S(\omega_1, \omega_2)$). □

Remark A.10. Let $\alpha\colon \mathbf{T} \to \mathbf{T}'$ be a tensor functor of Tannakian categories over k_0. If ω and ω' are fibre functors of $\mathbf{T}$ and $\mathbf{T}'$ respectively over S and $\omega = \omega' \circ \alpha$, then α defines a morphism of S/S_0-groupoids $Aut^{\otimes}_{k_0}(\omega') \to Aut^{\otimes}_{k_0}(\omega)$. When we drop the condition that $\omega = \omega' \circ \alpha$, then all we can say is that α defines a morphism $Aut^{\otimes}_{k_0}(\omega')_{S'} \to Aut^{\otimes}_{k_0}(\omega)_{S'}$ for some S' faithfully flat over S, and that this morphism is uniquely determined up to isomorphism.

Example A.11. Suppose $(\mathbf{T}, \otimes)$ has a fibre functor ω over k_0, i.e., that it is neutral. Then the groupoid $Aut^{\otimes}(\omega)$ is an affine group scheme G_0 over k_0, and ω defines an equivalence of $\mathbf{T}$ with the category of $\mathbf{Rep}_{k_0}(G_0)$ of finite-dimensional representations of G_0 over k_0.

Extension of scalars for Tannakian categories. For any category $\mathbf{T}$, one can define a category $\mathrm{Ind}(\mathbf{T})$ whose objects are the small filtered direct systems of objects in $\mathbf{T}$, and whose morphisms are given by

$$\mathrm{Hom}((X_\alpha), (Y_\beta)) = \varprojlim_{\alpha} \varinjlim_{\beta} \mathrm{Hom}(X_\alpha, Y_\beta).$$

Assume $\mathbf{T}$ is an abelian category whose objects are Noetherian (for example, a Tannakian category). Then $\mathbf{T}$ is a full subcategory of $\mathrm{Ind}(\mathbf{T})$, limits of small filtered direct systems in $\mathrm{Ind}(\mathbf{T})$ exist and are exact, and every object of $\mathrm{Ind}(\mathbf{T})$ is the limit of such a system of objects of $\mathbf{T}$. Conversely, these conditions determine $\mathrm{Ind}(\mathbf{T})$ uniquely up to a unique equivalence of categories (Deligne 1989, 4.2.2).

Let $(\mathbf{T}, \otimes)$ be a Tannakian category over k_0, and let ω be a fibre functor of $\mathbf{T}$ over a field k. Consider a diagram of fields

$$\begin{tikzcd} k \arrow[r] & k' \\ k_0 \arrow[u] \arrow[r] & k_0'. \arrow[u] \end{tikzcd}$$

Let X be an object of $\mathrm{Ind}(\mathbf{T})$ endowed with a homomorphism $i\colon k_0' \to \mathrm{End}_{k_0}(X)$ of k_0-algebras. We refer to the pair (X, i) as a ***k_0'-object*** in $\mathrm{Ind}(\mathbf{T})$. A subobject $Y \subset X$ ***generates*** (X,i) ***as a k_0'-object*** if it is not contained in any proper k_0'-object. Define $\mathbf{T} \otimes_{k_0} k_0'$ to be the category whose objects are the k_0'-objects of $\mathrm{Ind}(\mathbf{T})$ that are generated as k_0'-objects by a subobject in $\mathbf{T}$.

Proposition A.12. *Under the above assumptions, the category $\mathbf{T} \otimes_{k_0} k_0'$ is a Tannakian category over k_0', the fibre functor ω extends to a fibre functor ω' of $\mathbf{T} \otimes_{k_0} k_0'$ over k', and the k'/k_0'-groupoid $Aut^{\otimes}_{k_0'}(\omega')$ is the pull-back of the groupoid k/k_0-groupoid $Aut^{\otimes}_{k_0}(\omega)$.*

Proof. After (A.8), we may suppose that $\mathbf{T} = \mathbf{Rep}(S\colon \mathfrak{G})$, where $S = \operatorname{Spec} k$, and that ω is the forgetful functor. Then the statement follows from (Deligne 1989, 4.6iii). $\square$

Example A.13. Take $k_0' = k' = k$. The proposition then shows that $\mathbf{T} \otimes_{k_0} k$ is a neutral Tannakian category over k, that ω extends to a fibre functor ω' of $\mathbf{T} \otimes_{k_0} k$, and that the affine group scheme attached to $(\mathbf{T} \otimes_{k_0} k, \omega')$ is $\mathfrak{G}^{\Delta}$.

Gerbs. Recall that $\mathbf{Aff}_{S_0}$ is the category of affine schemes over S_0. A ***fibred category*** over $\mathbf{Aff}_{S_0}$ is a functor $p\colon \mathcal{F} \to \mathbf{Aff}_{S_0}$ such that every morphism $\alpha\colon T \to S$ in $\mathbf{Aff}_{S_0}$ defines an "inverse image" functor $\alpha^*\colon \mathcal{F}(S) \to \mathcal{F}(T)$ with certain natural properties (see Deligne and Milne 1982, p221). Here $\mathcal{F}(S)$ is the category $p^{-1}(S)$; it is called the ***fibre*** over S.

A fibred category is a ***pre-stack*** if for every pair of objects a, b of $\mathcal{F}(S)$, the functor sending an affine S-scheme $u\colon T \to S$ to $\mathrm{Hom}(u^*a, u^*b)$ is a sheaf for the faithfully flat topology on S (see Waterhouse 1979, 15.6, for the notion of a sheaf for the faithfully flat (= fpqc) topology). It is a ***stack*** if, for every faithfully flat morphism $T' \to T$ in $\mathbf{Aff}_{S_0}$, the natural functor sending an object of $\mathcal{F}(T)$ to an object of $\mathcal{F}(T')$ with a descent datum is an equivalence of categories (i.e., descent is ***effective*** on objects).

A stack is a ***gerb*** if it satisfies the following conditions:

A.14.1. it is nonempty;

A.14.2. the fibres are groupoids in sets;

A.14.3. any two objects are locally isomorphic.

Let $\mathcal{G}$ be a gerb over S_0, and let $e \in \mathcal{G}(S)$ for some nonempty S. Write $Aut(e)$ for the functor whose value on an affine S-scheme $c\colon T \to S$ is the set of automorphisms of c^*e regarded as an object of the category $\mathcal{G}(T)$. The gerb is said to be ***affine*** if this functor is representable by an affine group scheme over S (the group scheme then has the structure of a band over S_0).

Groupoids and gerbs. Let $\mathfrak{G}$ be an S/S_0-groupoid. By definition, for any S_0-scheme T, the quadruple $(S(T), \mathfrak{G}(T), (t,s), \circ)$ is a groupoid in sets. For varying T, these categories form a fibred category $\mathcal{G}^0(S\colon \mathfrak{G}) \to S$. It is a pre-stack. Let $\mathcal{G}(S\colon \mathfrak{G})$ be the stack associated with $\mathcal{G}^0$ for the faithfully flat topology. It contains $\mathcal{G}^0$ as a full subcategory, and it is characterized by having the property that any object of $\mathcal{G}$ is locally in $\mathcal{G}^0$.

Proposition A.15. *Let $\mathfrak{G}$ be an S_0-groupoid acting on a nonempty scheme S. The stack $\mathcal{G}(S\colon \mathfrak{G})$ is a gerb if and only if $\mathfrak{G}$ acts transitively on S.*

Proof. This is almost obvious—see [Deligne 1990, 3.3.] □

Let $\mathcal{G} = \mathcal{G}(S\colon \mathfrak{G})$. Then $\mathcal{G}(S)$ has a distinguished object, namely, the identity morphism of S. Let $\mathfrak{G}$ and $\mathfrak{G}'$ be S_0-groupoids acting transitively on S. A morphism $\alpha\colon \mathfrak{G} \to \mathfrak{G}'$ defines a morphism of fibred categories $\mathcal{G}^0(S\colon \mathfrak{G}) \to \mathcal{G}^0(S\colon \mathfrak{G}')$, and hence a morphism of gerbs $\mathcal{G}(S\colon \mathfrak{G}) \to \mathcal{G}(S\colon \mathfrak{G}')$, carrying the distinguished object of $\mathcal{G}(S\colon \mathfrak{G})(S)$ to that of $\mathcal{G}(S\colon \mathfrak{G}')(S)$.

Conversely, let $\mathcal{G}$ be an affine gerb over $\mathbf{Aff}_{S_0}$, and choose an object e of $\mathcal{G}(S)$ for some nonempty affine scheme S over S_0. For any $S \times_{S_0} S$-scheme $(b,a)\colon T \to S \times_{S_0} S$, let $Aut_{S_0}(e)(T)$ be the set of isomorphisms $a^*e \to b^*e$ (in the category $\mathcal{G}(T)$). This functor is represented by an S_0-groupoid $\mathfrak{G}$ acting transitively on S, which we call the ***groupoid of S_0-automorphisms*** of e.

These operations are inverse: if $\mathfrak{G}$ is an S_0-groupoid acting transitively on a nonempty affine scheme S over S_0, then $\mathcal{G}(S\colon \mathfrak{G})$ is an affine gerb with distinguished element $e = \mathrm{id}_S$ in $\mathcal{G}(S)$ and $Aut_{S_0}(e) = \mathfrak{G}$; if $\mathcal{G}$ is an affine gerb over S_0 and e

is an object of $\mathcal{G}(S)$, then $Aut_{S_0}(e)$ is represented by an S_0-groupoid $\mathfrak{G}$ acting transitively on S, and there is a canonical fully faithful functor $\mathcal{G}^0(S\colon \mathfrak{G}) \to \mathcal{G}$ which induces an equivalence of gerbs $\mathcal{G}(S\colon \mathfrak{G}) \to \mathcal{G}$. (See Deligne 1990, 3.4.)

The classification of groupoids. For a band G, the cohomology set $H^2(S_0, G)$ is defined to be the set of G-equivalence classes of gerbs over S_0 bound by G. We define the cohomology class of an S/S_0-groupoid $\mathfrak{G}$ to be the cohomology class of the associated gerb $\mathcal{G}(S\colon \mathfrak{G})$. When $\mathfrak{G}$ is commutative, this definition can be made more explicit (see below).

Proposition A.16. *Let $\mathfrak{G}$ and $\mathfrak{G}'$ be commutative S/S_0-groupoids, and let $\varphi\colon \mathfrak{G}^\Delta \to \mathfrak{G}'^\Delta$ be a homomorphism of commutative group schemes over S_0; then φ extends to a morphism of gerbs if and only if it maps the cohomology class of $\mathfrak{G}$ to that of $\mathfrak{G}'$.*

Proof. After replacing $\mathfrak{G}$ with $\varphi_*\mathfrak{G}$, we can assume that φ is the identity map, in which case the proposition is obvious. □

Morphisms of groupoids. Let $\mathfrak{P}$ and $\mathfrak{G}$ be transitive S/S_0-groupoids, and let φ and ψ be morphisms $\mathfrak{P} \to \mathfrak{G}$. For any S-scheme T, $\varphi(T)$ and $\psi(T)$ are homomorphisms of groupoids in sets. Define $Isom(\varphi, \psi)$ to be the subscheme of G such that, for any S-scheme T, $Isom(\varphi,\psi)(\mathrm{T})$ is the set of isomorphisms $\varphi(T) \to \psi(T)$. A section of $\mathfrak{P}$ over $S \times_{S_0} S$ defines a descent datum on $Isom(\varphi, \psi)$ which is independent of the choice of the section. Therefore $Isom(\varphi,\psi)$ is defined over k_0. Let $Aut(\varphi) = Isom(\varphi, \varphi)$. Then $Aut(\varphi)$ is a group scheme over k_0, and $Isom(\varphi, \psi)$ is either empty or is a right $Aut(\varphi)$-torsor. If $S = \operatorname{Spec} k_0^{\mathrm{al}}$, then

$$Isom(\varphi,\psi)(k_0) = \{g \in G(k_0^{\mathrm{al}}) \mid \operatorname{ad} g \circ \varphi = \psi\}.$$

Example A.17. As we noted in (A.6), there is a canonical morphism $\varphi\colon \mathfrak{G}_0 \to \mathfrak{G}_{G_0}$ from the trivial gerb to the neutral gerb defined by a group scheme G_0 over k_0. For this morphism

$$Aut(\varphi) = G_0.$$

Proposition A.18. *Let $\varphi_0\colon \mathfrak{P} \to \mathfrak{G}$ be a homomorphism of k_0^{al}/k_0-groupoids, and assume that $\mathfrak{P}$ is commutative and that the kernels of $\mathfrak{P}$ and $\mathfrak{G}$ are of finite type. The S_0-scheme $Isom(\varphi_0, \varphi)$ is nonempty (and hence an $Aut(\varphi_0)$-torsor) if and only φ_0^Δ is conjugate to φ^Δ by an element of $G(k_0^{\mathrm{al}})$. The map sending φ to the cohomology class of the torsor $Isom(\varphi_0, \varphi)$ defines a bijection from the set of isomorphism classes of homomorphisms $\varphi\colon \mathfrak{P} \to \mathfrak{G}$ such that φ^Δ and φ_0^Δ are conjugate to $H^1(S/S_0, Aut(\varphi_0))$.*

Proof. Omitted. □

The extension defined by a groupoid. Suppose S is Galois over S_0. By definition, this means that there is a profinite group Γ acting on S over S_0 such that the map

$$S \times \Gamma \to S \times_{S_0} S, \quad (s, \gamma) \mapsto (s, s \cdot \gamma)$$

is an isomorphism of schemes. Here Γ is to be interpreted as a finite or pro-finite (hence affine) scheme over S.

Now assume that $S = \operatorname{Spec} k$ with $k = k_0^{\mathrm{al}}$, and that there is a section of $\mathfrak{G}$ over $S \times_{S_0} S$. We can use the above isomorphism to identify $(S \times_{S_0} S)(S)$ with $\Gamma = \mathrm{Gal}(k/k_0)$, and so the map $\mathfrak{G} \to S \times_{S_0} S$ defines a surjection $\mathfrak{G}(k) \to \mathrm{Gal}(k/k_0)$. There is a unique way of putting a group structure on $\mathfrak{G}(k)$ so that

$$0 \longrightarrow G(k) \longrightarrow \mathfrak{G}(k) \longrightarrow \mathrm{Gal}(k/k_0) \longrightarrow 0$$

is an exact sequence of groups (here $G = \mathfrak{G}^{\Delta}$). Thus a k/k_0-groupoid can be thought of as an extension as above with additional structure. This is the approach adopted in [Langlands and Rapoport 1987].

In the case that G is commutative, the cohomology class of $\mathfrak{G}$ is the class attached in the usual way to the above exact sequence, i.e., if for a suitable section s to the map $\mathfrak{G}(k) \to \Gamma$, we write $s(\rho) \cdot s(\tau) = d_{\rho,\tau} s(\rho\tau)$, then $(\rho, \tau) \mapsto d_{\rho,\tau}$ is a 2-cocycle representing the class of $\mathfrak{G}$ in $H^2(k_0, G_0)$.

Remark A.19. Let $\mathfrak{P}$ be a commutative k_0^{al}/k_0-groupoid with kernel P_0. Let $\varphi_0 \colon P_0 \to G_0$ be a homomorphism of group schemes over k_0, and let Z_{φ_0} be the centralizer of $\varphi_0(P_0)$ in G_0. Assume φ_0 extends to a homomorphism $\varphi \colon \mathfrak{P} \to \mathfrak{G}_{G_0}$, and let $I_\varphi = Aut(\varphi)$. Then I_φ is an inner form of Z_{φ_0} whose cohomology class can be described as follows. Choose a suitable section s, as above, and let $(d_{\rho,\tau})$ be the corresponding 2-cocycle. When we write $\varphi(s(\rho)) = (c_\rho, \rho)$, we obtain a 1-cochain (c_ρ) splitting the cocycle $(\varphi(d_{\rho,\tau}))$:

$$c_\rho \cdot \rho c_\tau = \varphi(d_{\rho,\tau}) \cdot c_{\rho\tau}.$$

For $p \in P_0(k_0^{\mathrm{al}})$ we have

$$\rho\varphi_0(p) = \varphi_0(\rho p) = \varphi_0(s(\rho) \cdot p \cdot s(\rho)^{-1}) = (c_\rho, \rho) \cdot \varphi_0(p) \cdot (c_\rho, \rho)^{-1} = c_\rho \cdot \rho\varphi_0(p) \cdot c_\rho^{-1},$$

and so $c_\rho \in Z_{\varphi_0}(k_0^{\mathrm{al}})$. The formula displayed above shows that the image of (c_ρ) in $Z_{\varphi_0}/\varphi_0(P)$ is a cocycle. Its class in $H^1(k_0, Z_{\varphi_0}/\varphi_0(P))$ depends only on the isomorphism class of φ, and it is the cohomology class of I_φ.

The classification of Tannakian categories in terms of gerbs. Let $\mathbf{T}$ be a Tannakian category over k_0. For any affine scheme T over k_0, let $\mathrm{FIB}(\mathbf{T})(T)$ be the category of fibre functors over T. Then $\mathrm{FIB}(\mathbf{T})$ is in a natural way a fibred category over $\mathbf{Aff}_{k_0}$.

Theorem A.20. *Let* $\mathbf{T}$ *be a Tannakian category over* k_0*; then the fibred category* $\mathrm{FIB}(\mathbf{T})$ *is a gerb over* S_0*, and the obvious tensor functor*

$$\mathbf{T} \to \mathbf{Rep}(\mathrm{FIB}(\mathbf{T}))$$

is an equivalence of Tannakian categories. If $\mathbf{T}'$ *is a second Tannakian category, then the functor*

$$\mathbf{Hom}(\mathbf{T}, \mathbf{T}') \to \mathbf{Hom}(\mathrm{FIB}(\mathbf{T}), \mathrm{FIB}(\mathbf{T}'))$$

is an equivalence of categories.

Proof. The proof of this in (Saavedra 1972, III) becomes valid once (A.9) is acquired. □

Notes. For a survey of groupoids in sets, see [Brown 1987]. There is no good detailed account of groupoids in schemes, but [Deligne 1989] and [Deligne 1990] contain summaries. The theory of Tannakian categories is scattered among [Saavedra 1972], [Deligne and Milne 1982], and [Deligne 1990].

APPENDIX B. THE COHOMOLOGY OF REDUCTIVE GROUPS

In this section we review some results in the Galois cohomology of reductive groups. Throughout, k is a field of characteristic zero, and k^{al} is an algebraic closure of k. Reductive groups are assumed to be connected.

Inner forms. Let G be an algebraic group over k. An ***inner automorphism*** of G is an automorphism defined by an element of $G^{\mathrm{ad}}(k)$. An ***inner form*** of G is a pair (I, α) consisting of an algebraic group I over k and a $G(k^{\mathrm{al}})$-conjugacy class of isomorphisms $a\colon G_{k^{\mathrm{al}}} \to I_{k^{\mathrm{al}}}$ such that $a^{-1} \circ \tau a$ is an inner automorphism of $G_{k^{\mathrm{al}}}$ for all $\tau \in \mathrm{Gal}(k^{\mathrm{al}}/k)$. Two inner forms (I, α) and (I', α') are said to be ***isomorphic*** if there is an isomorphism of algebraic groups $\varphi\colon I \to I'$ (over k) such that

$$a \in \alpha \Rightarrow \varphi \circ a \in \alpha'.$$

Note that φ is then uniquely determined up to an inner automorphism of G over k. An inner form is said to be ***trivial*** if it is isomorphic to (G, id). When (I, α)

is an inner form of G and $a \in \alpha$, we often loosely refer to $a\colon G_{k^{\mathrm{al}}} \to I_{k^{\mathrm{al}}}$ as ***inner twisting*** of G, and (even more loosely) we write $a\colon G \to I$.

If (I, α) is an inner form of G and $a \in \alpha$, then $c_\tau = a^{-1} \circ \tau a$ is a 1-cocycle for G^{ad} whose cohomology class does not depend on the choice of a in α. In this way the set of isomorphism classes of inner forms of G becomes identified with $H^1(k, G^{\mathrm{ad}})$.

The algebraic fundamental group. Let G be a reductive group over a field k. Let G^{sc} be the simply connected covering group of G^{der}, and let $\rho\colon G^{\mathrm{sc}} \to G$ be the composite

$$G^{\mathrm{sc}} \twoheadrightarrow G^{\mathrm{der}} \hookrightarrow G.$$

Let T be a maximal torus in $G_{k^{\mathrm{al}}}$, and let $T^{\mathrm{sc}} = \rho^{-1}(T)$; it is a maximal torus in $G^{\mathrm{sc}}_{k^{\mathrm{al}}}$. The restriction of ρ is a homomorphism $T^{\mathrm{sc}} \to T$ with finite kernel, and we define

$$\pi_1(G, T) = X_*(T)/\rho_* X_*(T^{\mathrm{sc}}).$$

It is a finitely generated abelian group.

If T' is a second maximal torus in $G_{k^{\mathrm{al}}}$, then there exists a $g \in G(k^{\mathrm{al}})$ such that $T' = gTg^{-1}$.

Lemma B.1. *The map $\pi_1(G, T) \to \pi_1(G, T')$ induced by $\operatorname{ad} g$ is independent of the choice of g.*

Proof. See [Borovoi 1989/90, 1.2.] □

We let $\pi_1(G) = \pi_1(G, T)$ for any maximal torus T in $G_{k^{\mathrm{al}}}$, and we call it the ***algebraic fundamental group*** of G. According to the lemma, it is well-defined up to a canonical isomorphism. There is a natural action of $\Gamma =_{df} \operatorname{Gal}(k^{\mathrm{al}}/k)$ on $\pi_1(G)$: for example, if we choose T to be a maximal torus in G (rather than $G_{k^{\mathrm{al}}}$), then the action is the natural action of Γ on $X_*(T)/\rho X_*(T^{\mathrm{sc}})$.

Properties B.2.

(a) The algebraic fundamental group is an ***exact*** functor from the category of reductive groups over k to the category of $\operatorname{Gal}(k^{\mathrm{al}}/k)$-modules.

(b) For a torus T over k, $\pi_1(T) = X_*(T)$.

(c) For a semisimple group G, $\pi_1(G) = (\operatorname{Ker} \rho) \otimes \widehat{\mathbb{Z}}(-1)$. Here $\widehat{\mathbb{Z}}(1) =_{df} \varprojlim \mu_n(k^{\mathrm{al}})$ and $\widehat{\mathbb{Z}}(-1)$ is its dual.

(d) If G^{der} is simply connected, then $\pi_1(G) = \pi_1(G^{\mathrm{ab}}) = X_*(G^{\mathrm{ab}})$; in general there is an exact sequence

$$1 \to \operatorname{Ker} \rho \otimes \widehat{\mathbb{Z}}(-1) \to \pi_1(G) \to X_*(G^{\mathrm{ab}}) \to 1.$$

(e) An inner twisting $a\colon G_{k^{\mathrm{al}}} \to G'_{k^{\mathrm{al}}}$ of G induces an isomorphism $\pi_1(G) \to \pi_1(G')$.

(f) Let $G^\vee$ be the dual group of G; then there is a canonical isomorphism $\pi_1(G) \to X^*(Z(G^\vee))$ (see B.28 below).

(g) When $k = \mathbb{C}$, the topological fundamental group of $G(\mathbb{C})$ is equal to $\pi_1(G)$ (the isomorphism implicit in this statement depends on a choice of $\sqrt{-1}$).

(h) The étale fundamental group $\pi_1^{\mathrm{et}}(G)$ of G is equal to $\pi_1(G) \otimes \widehat{\mathbb{Z}}(1)$.

Example B.3. Let G be the quasi-split unitary group attached to a quadratic imaginary extension E of a totally real field F as, for example, in §6 of [Gordon 1991]. Then G^{der} is simply connected, and $G^{\mathrm{ab}} = (\mathbb{G}_m)_{E/\mathbb{Q}}$. Therefore

$$\pi_1(G) = X_*(\mathbb{G}_m)_{E/\mathbb{Q}} = \mathbb{Z}[\mathrm{Hom}(E, \mathbb{Q}^{\mathrm{al}})].$$

The functor $G \mapsto A(G)$. If M is a Γ-module, we define M^Γ and M_Γ respectively to be the largest submodule of M and the largest quotient module on which the action of Γ is trivial. For a reductive group G over k, define

$$A(G) = \pi_1(G)_{\Gamma,\mathrm{tors}}, \qquad \Gamma = \mathrm{Gal}(k^{\mathrm{al}}/k).$$

Then $G \mapsto A(G)$ is a functor from the category of reductive groups over k to the category of finite abelian groups.

Recall [Serre 1962, VIII.1] that, for a module M over a finite group Γ, the Tate cohomology group $H_T^{-1}(\Gamma, M)$ is defined to be the quotient of the kernel of the norm map

$$x \mapsto \textstyle\sum_{\tau\in\Gamma} \tau x \colon M \to M$$

by $I_\Gamma M$ where I_Γ is the ideal in $\mathbb{Z}[\Gamma]$ generated by the elements $\tau - 1$, $\tau \in \Gamma$.

Proposition B.4. *Let k be a field having extensions of arbitrarily large degrees. For any sufficiently large finite Galois extension k' of k,*

$$A(G) = H_T^{-1}(\mathrm{Gal}(k'/k), \pi_1(G)).$$

Proof. Let k' be a finite Galois extension of k splitting G, and let $\Gamma' = \mathrm{Gal}(k'/k)$. Then $\pi_1(G)_\Gamma = \pi_1(G)_{\Gamma'} = \pi_1(G)/I_{\Gamma'}\pi_1(G)$, and so there is an obvious inclusion

$$H_T^{-1}(\Gamma', \pi_1(G)) \hookrightarrow \pi_1(G)_\Gamma.$$

Because $\pi_1(G)$ is finitely generated, $H_T^{-1}(\Gamma', \pi_1(G))$ is finite (ibid. p138), and so the image of the map is contained in $\pi_1(G)_{\Gamma,\mathrm{tors}}$. Conversely, let x be a torsion element of $\pi_1(G)_\Gamma$, and write N for the norm map

$$\pi_1(G) \to \pi_1(G), \quad x \mapsto \sum_{\tau\in\Gamma} \tau x.$$

For some m, $mx \in I_{\Gamma'}\pi_1(G) \subset \mathrm{Ker}(N)$, and so $m \cdot Nx = 0$. After replacing k' by a suitable larger extension, we will have $x \in \mathrm{Ker}(N)$. □

Remark B.5.

(a) In terms of the dual group $G^\vee$, $A(G)$ is the dual of the finite group $\pi_0(Z(G^\vee)^\Gamma)$. Indeed,

$$(\pi_0(Z(G^\vee)^\Gamma)^{\mathrm{dual}} = (X^*(Z(G^\vee)^\Gamma))_{\mathrm{tors}} = (X^*(Z(G^\vee)))_{\Gamma,\mathrm{tors}}.$$

(b) If G is the unitary group in (B.3), then

$$A(G) = H_T^1(K/k, X_*(T)) = 0.$$

Crossed modules. Let Γ be either a discrete or profinite group. By a Γ-***module*** we mean a group N (not necessarily commutative) together with a continuous action

$$\Gamma \times N \to N$$

of Γ on N. Since N is endowed with the discrete topology, the continuity condition is vacuous if Γ is discrete, and it means that

$$N = \cup N^{\Gamma'} \text{ (union over open subgroups } \Gamma' \text{ of } \Gamma)$$

when Γ is profinite.

Definition B.6. A ***crossed module*** is a homomorphism of groups

$$M \xrightarrow{\alpha} N$$

together with a left action of N on M (denoted by $(n, m) \mapsto {}^n m$) such that:

(B.6.1) for $m, m' \in M$, ${}^{\alpha(m)}m' = (\mathrm{ad}\, m)(m')$;

(B.6.2) for $m \in M$ and $n \in N$, $\alpha({}^n m) = (\mathrm{ad}\, n)\alpha(m)$.

A Γ-***action*** on a crossed module $(M \to N)$ is a continuous action of Γ on M and N such that the maps

$$M \xrightarrow{\alpha} N, \qquad N \to \mathrm{Aut}(M)$$

commute with the action of Γ.

The crossed module $\alpha\colon M \to N$ is to be regarded as a very short complex, with M in the -1 position and N in the 0 position. Henceforth, by a crossed module, we shall always mean a crossed module with Γ-action.

Example B.7.

(a) For any Γ-module N, there is a crossed module $1 \to N$. For any ***abelian*** Γ-module M, there is a crossed module $M \to 1$. We usually write 1 for the crossed module $1 \to 1$.

(b) If M is a normal subgroup of the Γ-module N that is stable under the action of Γ, then the inclusion map $M \to N$ becomes a crossed module with the action ${}^n m = nmn^{-1}$.

(c) Any surjective homomorphism $M \to N$ of Γ-modules with central kernel becomes a crossed module with the natural action of N on M.

(d) If M and N are commutative, then any homomorphism $M \to N$ of Γ-modules can be regarded as a crossed module with N acting trivially on M. A crossed module of this form is said to be ***commutative.***

(e) For any Γ-module M, $m \mapsto \mathrm{ad}\, m\colon M \to \mathrm{Aut}(M)$ is a crossed module.

(f) For any reductive group G over k, the map $\rho\colon G^{\mathrm{sc}} \to G$ defines a crossed module $G^{\mathrm{sc}}(k^{\mathrm{al}}) \to G(k^{\mathrm{al}})$ over $\Gamma = \mathrm{Gal}(k^{\mathrm{al}}/k)$.

Lemma B.8. *Let $\alpha\colon M \to N$ be a crossed module.*

(a) *The group $\mathrm{Ker}(\alpha)$ is central in M, and is invariant under the action of N.*

(b) *The group $\mathrm{Im}(\alpha)$ is normal in N.*

Proof. Both statements follow directly from the definition of crossed module. □

Definition B.9. A ***homomorphism*** of crossed modules

$$\varepsilon\colon (M_1 \xrightarrow{\alpha_1} N_1) \to (M_2 \xrightarrow{\alpha_2} N_2)$$

is a pair of homomorphisms of Γ-modules $(\varepsilon_{-1}\colon M_1 \to M_2,\ \varepsilon_0\colon N_1 \to N_2)$ such that

$$\begin{array}{ccc} M_1 & \xrightarrow{\varepsilon_{-1}} & M_2 \\ {\scriptstyle\alpha_1}\downarrow & & \downarrow{\scriptstyle\alpha_2} \\ N_1 & \xrightarrow{\varepsilon_0} & N_2 \end{array}$$

commutes and

$$\varepsilon_{-1}({}^{n}m) = {}^{\varepsilon_0(n)}\varepsilon_{-1}(m) \text{ for all } n \in N_1, \quad m \in M_1.$$

A homomorphism ε of crossed modules is said to be a ***quasi-isomorphism*** if the homomorphisms

$$H^{-1}(\varepsilon)\colon \operatorname{Ker}(\alpha_1) \to \operatorname{Ker}(\alpha_2); \quad H^0(\varepsilon)\colon \operatorname{Coker}(\alpha_1) \to \operatorname{Coker}(\alpha_2)$$

are isomorphisms.

Example B.10.

(a) If α is injective, then the homomorphism of crossed modules

$$(M \xrightarrow{\alpha} N) \to (1 \xrightarrow{\alpha} N/\alpha(M))$$

is a quasi-isomorphism.

(b) If α is surjective, then the homomorphism of crossed modules

$$(\operatorname{Ker}\alpha \to 1) \to (M \to N)$$

is a quasi-isomorphism.

(c) For any maximal torus T in the reductive group G,

$$(T^{\mathrm{sc}}(k^{\mathrm{al}}) \xrightarrow{\rho} T(k^{\mathrm{al}})) \to (G^{\mathrm{sc}}(k^{\mathrm{al}}) \xrightarrow{\rho} G(k^{\mathrm{al}}))$$

is a quasi-isomorphism.

Definition B.11. A sequence of homomorphisms of crossed modules

$$1 \to (M_1 \to N_1) \xrightarrow{\varepsilon} (M_2 \to N_2) \xrightarrow{\varepsilon'} (M_3 \to N_3) \to 1$$

is said to be ***exact*** if the sequences

$$1 \to M_1 \to M_2 \to M_3 \to 1, \quad 1 \to N_1 \to N_2 \to N_3 \to 1$$

are exact.

Example B.12. For any crossed module $(M \xrightarrow{\alpha} N)$ there is an exact sequence

$$1 \to (1 \to N) \to (M \to N) \to (M \to 1) \to 1.$$

The cohomology of crossed modules. It is possible to define cohomology sets $\mathbb{H}^i(\Gamma, M \to N)$ for $i = -1, 0, 1$.

Definition B.13.

(a) Set

$$\mathbb{H}^{-1}(\Gamma, M \to N) = \mathrm{Ker}(\alpha)^\Gamma;$$

it is an abelian group.

(b) Write $\mathrm{Maps}(\Gamma, M)$ for the set of maps $\varphi\colon \Gamma \to M$, and set

$$C^0 = \mathrm{Maps}(\Gamma, M) \times N$$

$$Z^0 = \left\{(\varphi, n) \in C^0 \mid \varphi(\sigma\tau) = \varphi(\sigma) \cdot {}^\sigma\varphi(\tau),\ \ {}^\sigma n = \alpha(\varphi(\sigma)^{-1}) \cdot n,\ \ \sigma, \tau \in \Gamma\right\}.$$

The set C^0 has a group structure

$$(\varphi_1, n_1) \cdot (\varphi_2, n_2) = ({}^{n_1}\varphi_2 \cdot \varphi_1, n_1 n_2)$$

for which Z^0 is a subgroup. The map

$$\nu\colon M \to Z^0, \quad m \mapsto (\varphi, \alpha(m)), \quad \varphi(\sigma) = m \cdot {}^\sigma m^{-1}$$

is a homomorphism whose image is a normal subgroup of Z^0. Define

$$\mathbb{H}^0(\Gamma, M \to N) = Z^0/\nu(M).$$

It is a group.

(c) The set Z^1 of 1-cocycles is defined to be the subset of

$$C^1 =_{df} \mathrm{Maps}(\Gamma \times \Gamma, M) \times \mathrm{Maps}(\Gamma, N)$$

of pairs (h, ψ) such that, for $\sigma, \tau, \upsilon \in \Gamma$,

$$\alpha(h(\sigma, \tau)) \cdot \psi(\sigma\tau) = \psi(\sigma) \cdot {}^\sigma\psi(\tau)$$
$${}^{\psi(\sigma)\sigma}h(\tau, \upsilon) \cdot h(\sigma, \tau\upsilon) = h(\sigma, \tau) \cdot h(\sigma\tau, \upsilon).$$

There is a natural right action of C^0 on Z^1, namely, for $(a, n) \in C^0$,

$$(h, \psi) * (a, n) = (h', \psi')$$

where

$$\psi'(\sigma) = n^{-1} \cdot \alpha(a(\sigma)) \cdot \psi(\sigma) \cdot {}^\sigma n$$
$$h'(\sigma, \tau) = {}^{n^{-1}}\left(a(\sigma) \cdot {}^{\psi(\sigma)\sigma}a(\tau) \cdot h(\sigma, \tau) \cdot a(\sigma\tau)^{-1}\right).$$

We define

$$\mathbb{H}^1(\Gamma, M \to N) = Z^1/C^0.$$

It is a set with a distinguished ***neutral*** element, namely, that represented by the trivial cocycle (1,1).

Properties B.14.

(a) $\mathbb{H}^0(1 \to N) = H^0(\Gamma, N)$; $\mathbb{H}^0(M \to 1) = H^1(\Gamma, M)$.

(b) $\mathbb{H}^1(1 \to N) = H^1(\Gamma, N)$; $\mathbb{H}^1(M \to 1) = H^2(\Gamma, M)$ (which is defined, because M is commutative).

(c) If $M \to N$ is a commutative crossed module, then $\mathbb{H}^i(M \to N)$ is the usual hypercohomology of the complex $M \to N$.

(d) [Borovoi 1991, 2.16.] A short exact sequence of crossed modules

$$1 \to (M_1 \to N_1) \xrightarrow{\varepsilon} (M_2 \to N_2) \xrightarrow{\varepsilon'} (M_3 \to N_3) \to 1$$

gives rise to an exact sequence

$$1 \to \mathbb{H}^{-1}(\Gamma, M_1 \to N_1) \to \mathbb{H}^{-1}(\Gamma, M_2 \to N_2) \to \cdots \to \mathbb{H}^1(\Gamma, M_3 \to N_3).$$

For example, from (B.12) we see that, for any crossed module $(M \to N)$, there is an exact sequence

$$\cdots \to H^i(\Gamma, M) \to H^i(\Gamma, N) \to \mathbb{H}^i(\Gamma, M \to N) \to \ldots.$$

(e) (Ibid. 2.22.) Suppose in the above short exact sequence that $(M_1 \to N_1)$ is ***central*** in $(M_2 \to N_2)$, i.e., M_1 is central in M_2, N_1 is central in N_2, and N_2 acts trivially on M_1. Then the sequence extends to an exact sequence:

$$\cdots \to \mathbb{H}^1(M_2 \to N_2) \to \mathbb{H}^1(M_3 \to N_3) \to \mathbb{H}^2(M_1 \to N_1).$$

(f) (Ibid. 3.3.) The maps on cohomology induced by a quasi-isomorphism of crossed modules are bijections.

Example B.15. Let G be a reductive group over k. The quasi-isomorphism in (B.10c) defines an isomorphism $\mathbb{H}^i(k, T^{\mathrm{sc}} \to T) \to \mathbb{H}^i(k, G^{\mathrm{sc}} \to G)$ for each i. In particular, we see that these sets are commutative groups. The Tate-Nakayama isomorphisms sometimes allow us to compute $\mathbb{H}^*(k, T^{sc} \to T)$ in terms of $\mathbb{H}^*(k, X_*(T^{\mathrm{sc}}) \to X_*(T)) = H^*(k, \pi_1(G))$.

The map $G(k) \to \pi_1(G)^\Gamma$. Let k be a finite extension of $\mathbb{Q}_p$, and let $\Gamma = \mathrm{Gal}(k^{\mathrm{un}}/k)$. For any unramified torus T over k, there is a surjective homomorphism

$$T(k) \to X_*(T)^\Gamma = \pi_1(T)^\Gamma \tag{B.15.1}$$

obtained by tensoring the normalized valuation ord: $(k^{\mathrm{un}})^\times \twoheadrightarrow \mathbb{Z}$ with $X_*(T)$ and taking invariants under Γ. Now consider the two functors $G \mapsto G(k)$ and $G \mapsto \pi_1(G)^\Gamma$ from the category of unramified reductive groups over k to the category of groups.

Proposition B.16. *There exists a unique extension of (B.15.1) to a homomorphism of functors*

$$\lambda_G \colon G(k) \to \pi_1(G)^\Gamma.$$

For all G, the homomorphism λ_G is surjective, and every hyperspecial subgroup of $G(k)$ is contained in the kernel of λ_G.

Proof. Choose a Borel subgroup B of G defined over k, and let T be an unramified maximal torus of G contained in B. Consider first the diagram

$$\begin{tikzcd}
G^{\mathrm{sc}}(k) \arrow[r] & G(k) \arrow[r] & \mathbb{H}^0(k, G^{\mathrm{sc}} \to G) \arrow[d, equal] & \\
& H^0(k,T) \arrow[r] \arrow[d] & \mathbb{H}^0(k, T^{\mathrm{sc}} \to T) \arrow[r] \arrow[d] & H^1(k, T^{\mathrm{sc}}) \arrow[d] \\
& H^0(k, X_*(T)) \arrow[r] & \mathbb{H}^0(k, X_*(T^{\mathrm{sc}}) \to X_*(T)) \arrow[r] \arrow[d, equal] & H^1(k, X_*(T^{\mathrm{sc}})) \\
& & H^0(k, \pi_1(G)) \arrow[r, equal] & \pi_1(G)^\Gamma
\end{tikzcd}$$

in which the vertical arrows are induced by ord. Because G^{sc} is simply connected, the set of fundamental weights is a basis for $X^*(T^{\mathrm{sc}})$, and since G, B, T are defined over k, Γ preserves the fundamental weights. It follows that T^{sc} is a product of tori of the form $(\mathbb{G}_m)_{F/k}$ for certain finite extensions F of k, and so the two cohomology groups at right are zero. The diagram now provides a surjective homomorphism $\lambda_G \colon G(k) \to \pi_1(G)^\Gamma$ whose kernel contains $G^{\mathrm{sc}}(k)$ and $\mathrm{Ker}(\lambda_T)$. Since a hyperspecial group K can be written

$$K = \rho(K^{\mathrm{sc}}) \cdot (T(k) \cap K)$$

with $K^{\mathrm{sc}} \subset G^{\mathrm{sc}}(k)$ [Kottwitz 1984b, 3.34], it is contained in the kernel of λ_G. $\square$

Fundamental tori. Let G be a reductive group over a field k of characteristic zero. A ***fundamental torus*** $T \subset G$ is a maximal torus of minimal k-rank. The maps

$$T \mapsto T^{\mathrm{sc}}, \quad T' \mapsto \rho(T') \cdot Z(T)^\circ$$

determine a one-to-one correspondence between the maximal tori in G and those in G^{sc}. Clearly, fundamental tori correspond to fundamental tori under this correspondence.

Proposition B.17. *Every semisimple group over a non-archimedean local field contains an anisotropic torus.*

Proof. See [Kneser 1965 , II, p271]. □

Lemma B.18. *Let T be a fundamental torus of a simply connected semisimple group G over a local field k; then $H^2(k,T)=0$.*

Proof. If k is non-archimedean, then T is anisotropic, and the Tate-Nakayama isomorphism

$$H^0_T(\mathrm{Gal}(k'/k), X_*(T)) \to H^2(\mathrm{Gal}(k'/k, T(k')),$$

which exists for every finite Galois extension k' of k (see B.21 below), shows that $H^2(k,T)=0$. If $k=\mathbb{R}$, then T is isomorphic to a product of compact torus with copies of $(\mathbb{G}_m)_{\mathbb{C}/\mathbb{R}}$ (see, for example, [Kottwitz 1986, 10.4], and so the result is obvious. □

Lemma B.19. *Let T be a fundamental torus of a reductive group G over $\mathbb{R}$; then the map $H^1(\mathbb{R},T)\to H^1(\mathbb{R},G)$ is surjective.*

Proof. See [Kottwitz 1986, 10.1]; also [Langlands and Rapoport 1987, 5.14]. □

Proposition B.20. *If k is a local field, then there is an exact sequence (of abelian groups)*

$$H^1(k,G^{\mathrm{sc}}) \to H^1(k,G) \to \mathbb{H}^1(k, G^{\mathrm{sc}}\to G) \to 0.$$

Proof. Choose T to be fundamental in G, and consider the commutative diagram:

$$\begin{tikzcd}
H^1(G^{\mathrm{sc}}) \arrow[r, "\rho"] & H^1(G) \arrow[r] & \mathbb{H}^1(G^{\mathrm{sc}}\to G) & \\
& H^1(T) \arrow[u] \arrow[r] & \mathbb{H}^1(T^{\mathrm{sc}}\to T) \arrow[u, "\approx"'] \arrow[r] & H^2(T^{\mathrm{sc}}).
\end{tikzcd}$$

From (B.18) we know that $H^2(T^{\mathrm{sc}})=0$, and this implies that $H^1(G)\to\mathbb{H}^1(G^{\mathrm{sc}}\to G)$ is surjective. □

The group $A(G)$ in the local case. We first recall the local version of the Tate-Nakayama isomorphism.

Proposition B.21. *Let k' be a finite Galois extension of a local field k, and let $\Gamma'=\mathrm{Gal}(k'/k)$. For any finitely generated torsion-free Γ'-module M, cup-product with the fundamental class in $H^2(\Gamma',k'^{\times})$ defines an isomorphism*

$$H^r_T(\Gamma',M)\to H^{r+2}_T(\Gamma', M\otimes k'^{\times})$$

for all integers r.

Proof. See [Serre 1962, IX.8]. □

Proposition B.22. *Let k be a local field of characteristic zero. For any reductive group G, there is a canonical homomorphism*

$$\alpha_G\colon H^1(k,G) \to A(G),$$

which is functorial in G. If k is nonarchimedean, then α_G is an isomorphism; if $k=\mathbb{R}$, then there is an exact sequence

$$H^1(\mathbb{R},G^{\mathrm{sc}}) \to H^1(\mathbb{R},G) \to A(G) \to \pi_1(G)$$

in which the last map is induced by $\iota+1\colon \pi_1(G)_{\Gamma(\infty)} \to \pi_1(G)$.

Proof. Assume first that k is nonarchimedean. From (B.4) we know that for all sufficiently large finite Galois extensions k' of k,

$$A(G) = H_T^{-1}(\mathrm{Gal}(k'/k),\pi_1(G)) = \mathbb{H}_T^{-1}(\mathrm{Gal}(k'/k), X_*(T^{\mathrm{sc}}) \to X_*(T))$$

for any maximal torus T in G. It follows from (B.21) that this last group is canonically isomorphic to $\mathbb{H}^1(\mathrm{Gal}(k'/k), T^{\mathrm{sc}}(k') \to T(k'))$. Since $A(G)$ does not depend on k', when we pass to inverse limit, we obtain the first of the following isomorphisms

$$A(G) \xrightarrow{\approx} \mathbb{H}^1(k, T^{\mathrm{sc}} \to T) \xrightarrow{\approx} \mathbb{H}^1(k, G^{\mathrm{sc}} \to G).$$

The second was noted in (B.15). From (B.20) and the fact that $H^1(k,G^{\mathrm{sc}})=0$, we know that

$$H^1(k,G) \to \mathbb{H}^1(k,G^{\mathrm{sc}} \to G)$$

is an isomorphism, and this completes the proof in the nonarchimedean case.

The proof in the archimedean case is similar.

The group $A(G)$ in the global case. Let k be a number field, and let k' be a finite Galois extension of k with Galois group Γ'. In this case, the Tate-Nakayama isomorphisms compare the cohomology of the sequence

$$1 \to k'^{\times} \to \mathbb{I}_{k'} \to C_L \to 1$$

with that of the simpler sequence

$$0 \to X \to Y \to \mathbb{Z} \to 0$$

where Y is the free abelian group generated by the primes of k' and X is the kernel of the map

$$\sum n_v v \mapsto \sum n_v\colon Y \to \mathbb{Z}.$$

An element τ of the group Γ' acts on Y according to the rule:

$$\tau(\sum n_v v) = \sum n_v(\tau v) = \sum n_{\tau^{-1}v} v.$$

Theorem B.23. *For any finitely generated torsion-free* Γ*-module, there is a commutative diagram*

$$\begin{array}{ccccccccc}
\cdots \longrightarrow & H^r_T(\Gamma, X\otimes M) & \longrightarrow & H^r_T(\Gamma, Y\otimes M) & \longrightarrow & H^r_T(\Gamma, \mathbb{Z}\otimes M) & \longrightarrow \cdots \\
& \downarrow & & \downarrow & & \downarrow & \\
\cdots \longrightarrow & H^{r+2}_T(\Gamma, L^\times\otimes M) & \longrightarrow & H^{r+2}_T(\Gamma, \mathbb{I}\otimes M) & \longrightarrow & H^{r+2}_T(\Gamma, C\otimes M) & \longrightarrow \cdots .
\end{array}$$

Proof. See [Tate 1966, p717]. □

Theorem B.24. *For any reductive group* G *over a number field* k*, there exists a canonical exact sequence*

$$H^1(k,G) \to \oplus_v H^1(k_v,G) \to A(G),$$

which is functorial in G.

Proof (sketch). Consider the commutative diagram

$$\begin{array}{ccc}
H^1(k,G^{\mathrm{sc}}) & \xrightarrow{\text{onto}} & \oplus_v H^1(k_v,G^{\mathrm{sc}}) \\
\downarrow & & \downarrow \\
H^1(k,G) & \longrightarrow & \oplus_v H^1(k_v,G) \\
\downarrow & & \downarrow \\
\mathbb{H}^1(k,G^{\mathrm{sc}}\to G) & \longrightarrow & \oplus_v \mathbb{H}^1(k_v,G^{\mathrm{sc}}\to G).
\end{array}$$

The cokernel of the middle horizontal arrow maps injectively into the cokernel of the bottom arrow. Using (B.15), we can replace the G's in the bottom row with T's, and then (B.23) allows us to compute the cokernel. □

The map $H^1(k_v,G) \to A(G)$ in the theorem is the composite of the map $\alpha_G\colon H^1(k_v,G) \to A(G_{k_v})$ with the obvious map $A(G_{k_v}) \to A(G)$.

The σ-conjugacy classes. Let B be the field of fractions of the Witt vectors over $\mathbb{F}$. For a reductive group G over $\mathbb{Q}_p$, we write $\mathbb{B}(G)$ for the set of σ-conjugacy classes in $G(B)$, i.e., $\mathbb{B}(G) = G(B)/\sim$, where $g \sim g'$ if $g' = t\cdot g\cdot \sigma t^{-1}$ for some $t \in G(B)$.

Proposition B.25. *Regard* X_* *and* $\mathbb{B}$ *as functors from the category of tori over* $\mathbb{Q}_p$ *to the category of groups. There is a unique morphism of functors* $\beta\colon X_* \to \mathbb{B}$ *such that* $\beta_{\mathbb{G}_m}$ *is the map*

$$X_*(\mathbb{G}_m) \to \mathbb{B}(\mathbb{G}_m), \quad \mu \mapsto [\mu(p^{-1})].$$

For all T, β_T induces an isomorphism

$$X_*(T)_\Gamma \to \mathbb{B}(T).$$

Proof. This is proved in [Kottwitz 1985, 3.5]. Here we recall only the proof of the uniqueness. First consider the torus $T =_{df} (\mathbb{G}_m)_{L/\mathbb{Q}_p}$ for some finite extension L of $\mathbb{Q}_p$. The norm map $L^\times \to \mathbb{Q}_p^\times$ defines a homomorphism $\mathrm{Nm}\colon T \to \mathbb{G}_m$, and if $\beta(T)$ exists, there will be a commutative diagram:

$$\begin{array}{ccc} X_*(T) & \xrightarrow{\beta(T)} & \mathbb{B}(T) \\ \downarrow{\scriptstyle \mathrm{Nm}} & & \downarrow{\scriptstyle \mathrm{Nm}} \\ X_*(\mathbb{G}_m) & \xrightarrow{\beta(\mathbb{G}_m)} & \mathbb{B}(\mathbb{G}_m). \end{array}$$

The right hand map Nm is an isomorphism, and so the diagram shows that $\beta(T) = \mathrm{Nm}^{-1} \circ \beta(\mathbb{G}_m) \circ \mathrm{Nm}$.

Now consider an arbitrary torus T, and let $\mu \in X_*(T)$. Choose a field $L \subset \mathbb{Q}_p^{\mathrm{al}}$ such that μ is defined over L. There is a unique element $\mu_0 \in X_*((\mathbb{G}_m)_{L/\mathbb{Q}_p})$ such that

$$< \tau, \mu_0 >= \begin{cases} 1 \text{ if } \tau \text{ is the given embedding of } L \text{ into } \mathbb{Q}_p^{\mathrm{al}}, \\ 0 \text{ otherwise.} \end{cases}$$

On applying $\mathrm{Res}_{L/\mathbb{Q}_p}$ to $\mu\colon \mathbb{G}_m \to T_L$ and composing with the norm map $\mathrm{Res}_{L/\mathbb{Q}_p} T_L \to T$, we obtain a homomorphism $\alpha\colon (\mathbb{G}_m)_{L/\mathbb{Q}_p} \to T$ such that $\alpha \circ \mu_0 = \mu$. Now the commutative diagram

$$\begin{array}{ccc} X_*((\mathbb{G}_m)_{L/\mathbb{Q}_p}) & \xrightarrow{\beta} & \mathbb{B}((\mathbb{G}_m)_{L/\mathbb{Q}_p}) \\ \downarrow{\scriptstyle X_*(\alpha)} & & \downarrow{\scriptstyle \mathbb{B}(\alpha)} \\ X_*(T) & \xrightarrow{\beta} & \mathbb{B}(T) \end{array}$$

determines the image of μ under $\beta(T)$.

Remark B.26.

(a) Our normalization of the maps $\beta(T)$ is the opposite of that of [Kottwitz 1985]—he specifies that $\beta(\mathbb{G}_m)$ sends μ to $[\mu(p)]$. Our convention seems to be forced on us by Deligne's convention that $\mu(z)$ acts on H^{mn} as z^{-m}, not z^m.

(b) The map $\beta(T)$ can be described as follows: choose a finite extension L of B splitting T; the image of a cocharacter μ of T is the σ-conjugacy class of $\mathrm{Nm}\,\mu(\pi)$, where π is a uniformizing parameter for L and Nm is the norm map $T(L) \to T(B)$. In particular, when T is unramified, $\beta(T)$ is simply the map $\mu \mapsto [\mu(p)^{-1}]$. (See [Kottwitz 1985, 3.5].

We can regard β^{-1} as a functorial isomorphism $\mathbb{B}(T) \to X_*(T)_\Gamma = \pi_1(T)_\Gamma$. Both $\mathbb{B}$ and $\pi_1(\cdot)_\Gamma$ are functors from the category of reductive groups over $\mathbb{Q}_p$ to the category of groups.

Proposition B.27. *There is a unique functorial map*

$$\mathbb{B}(G) \to \pi_1(G)_\Gamma$$

extending the map β^{-1} on tori.

Proof. Observe that $\mathbb{B}(G) = H^1(\Gamma, G(B))$ where Γ is the free abelian (discrete) group generated by σ. The extension from tori to reductive groups can be made as usual using the quasi-isomorphism (B.10c).

The dual group. We review part of the theory of the dual group; for more details, see [Borel 1979] or [Kottwitz 1984a]. Throughout this subsection, k is a field of characteristic zero, $\Gamma = \mathrm{Gal}(k^{\mathrm{al}}/k)$, and G is an arbitrary reductive group over k.

Let G be a connected reductive group over k^{al}. The choice of a pair $B \supset T$ with B a Borel subgroup of G and T a maximal torus, determines a ***based root datum*** $\Psi_0(G, B, T) = (X^*, \Delta, X_*, \Delta^\vee)$ in which $X^* = X^*(T)$, $X_* = X_*(T)$, and Δ (resp. $\Delta^\vee$) is the set of simple B-positive roots (resp. coroots) of T. For any other pair $B' \supset T'$, there is an inner automorphism γ of G such that $\gamma(T) = T'$ and $\gamma(B) = B'$. The isomorphism $\Psi_0(G, B, T) \to \Psi_0(G, B', T')$ defined by γ is independent of the choice of γ. We can therefore drop B and T from the notation. The ***inverse*** of $\Psi_0(G)$ is defined to be

$$\Psi_0(G)^\vee = (X_*, \Delta^\vee, X^*, \Delta).$$

A ***splitting*** of G is a triple $(B, T, \{X_\alpha\}_{\alpha \in \Delta^\vee})$ with X_α a nonzero element of the root space Lie $(G)_\alpha$.

Now assume that G is defined over k. We write $\Psi_0(G)$ for $\Psi_0(G_{k^{\mathrm{al}}})$. In this case, Γ acts on $\Psi_0(G)$. A connected reductive group $G^\vee$ over $\mathbb{C}$ together with an action of Γ will be called a ***dual group***[9] for G if $\Psi_0(G^\vee)$ is Γ-isomorphic to

[9]The usual notation is $\widehat{G}$ but $\widehat{\ }$ is better reserved for completions. Strictly, $G^\vee$ should be called the identity component of the L-group, and it should be denoted ${}^LG^0$.

$\Psi_0(G)^\vee$ and Γ preserves some splitting of $G^\vee$. For example, the dual group of a torus T is the torus $T^\vee$ over $\mathbb{C}$ such that $X_*(T^\vee) = X^*(T)$ with Γ acting on $T^\vee(\mathbb{C}) = X^*(T) \otimes \mathbb{C}^\times$ through its action on $X^*(T)$.

We shall mainly be concerned with the centre of the dual group. The following description of it will be useful: let $D = G^{\mathrm{ab}}$ and let $C = \pi_1^{\mathrm{al}}(G^{\mathrm{der}})$ (the centre of the simply connected covering group of G^{der}); then the identity component of $Z(G^\vee)$ is $D^\vee$, and the quotient of $Z(G^\vee)$ by $D^\vee$ is the dual C^{dual} of C. Therefore, there is an exact sequence

$$1 \to D^\vee \to Z(G^\vee) \to C^{\mathrm{dual}} \to 1.$$

In particular, $Z(G^\vee)$ is connected if and only if G^{der} is simply connected, in which case $Z(G^\vee) = D^\vee$.

A homomorphism $\gamma\colon G \to H$ of connected reductive groups is said to be ***normal*** if its image is a normal subgroup of H. Once splittings have been chosen for G and H, γ determines a homomorphism $\alpha^\vee : H^\vee \to G^\vee$. A change in the choice of the splittings does not affect $\gamma|Z(H^\vee)$, and we have a contravariant functor $G \mapsto Z(G^\vee)$ from the category of connected reductive groups over k and normal homomorphisms to the category of diagonalizable groups over $\mathbb{C}$ with an action of Γ. Furthermore, an exact sequence

$$1 \to G_1 \to G_2 \to G_3 \to 1$$

gives rise to an exact sequence

$$1 \to Z(G_3^\vee) \to Z(G_2^\vee) \to Z(G_3^\vee) \to 1.$$

Relation of $\pi_1(G)$ to the dual group.

Proposition B.28. *For any reductive group G, $\pi_1(G)$ and $X^*(Z(G^\vee))$ are canonically isomorphic.*

Proof. Let T be a maximal torus of G. Then there is a maximal torus $T^\vee \subset G^\vee$ such that $X^*(T^\vee) = X_*(T)$, and $R(G^\vee, T^\vee) = R^\vee(G,T)$ where R and $R^\vee$ denote the systems of roots and coroots respectively. Moreover

$$Z(G^\vee) = \bigcap_{\alpha^\vee \in R(G^\vee, T^\vee)} \mathrm{Ker}(\alpha^\vee\colon T^\vee \to \mathbb{G}_{m\mathbb{C}}).$$

Hence

$$X^*(Z(G^\vee)) = X^*(T^\vee)/ < R(G^\vee, T^\vee) >= X_*(T)/ < R^\vee > .$$

All the coroots $\alpha^\vee \in R^\vee \subset X_*(T)$ come from $X_*(T^{\mathrm{sc}})$, and the subset $R^\vee$ of $\rho_* X_*(T^{\mathrm{sc}})$ generates it. Therefore

$$X^*(Z(G)) = X_*(T)/\rho_* X_*(T^{\mathrm{sc}}) = \pi_1(G).$$

Notes. The definitions and results in B.1 through B.15 are taken from [(Borovoi 1989/90] and [Borovoi 1991]. Propositions B.16, B.22, B.24, B.27 are results of Kottwitz, [Kottwitz 1984b, 3.3; 1986, 1.2; 1986, 2.5; 1990, 6.1] respectively), except that, since he used the dual group in his proofs, Kottwitz only showed that the maps are functorial with respect to normal homomorphisms. The proofs given here are either in [Borovoi 1991] or are easy given Borovoi's methods and Kottwitz's original proofs. Proposition B.28 is from [Borovoi 1989/90].

APPENDIX C: RELATION TO THE TRACE ON THE INTERSECTION COHOMOLOGY GROUPS

In this appendix, I explain how the problem of computing the trace of a

$$\text{(Frobenius automorphism)} \times \text{(Hecke operator)}$$

on the intersection cohomology groups of a Shimura variety relates to the problem of describing the set of points of the Shimura variety with coordinates in the algebraic closure of a finite field, together with the actions of the Frobenius automorphism and the Hecke operators.

The Lefschetz trace formula. Let S be a smooth algebraic variety over an algebraically closed field k, and let $\mathcal{V}$ be a local system of $\mathbb{Q}_\ell$-vector spaces on S_{et}, some $\ell \neq \operatorname{char} k$. A ***correspondence*** on $(S, \mathcal{V})$ is a pair of mappings

$$S \xleftarrow{\alpha} T \xrightarrow{\beta} S$$

and a homomorphism $\gamma\colon \alpha^*\mathcal{V} \to \beta^*\mathcal{V}$. When β is finite, there is a canonical trace map $\beta_*\beta^*\mathcal{F} \to \mathcal{F}$, and consequently maps

$$H^i(S, \mathcal{V}) \to H^i(T, \alpha^*\mathcal{V}) \xrightarrow{\gamma} H^i(T, \beta^*\mathcal{V}) \xrightarrow{\text{trace}} H^i(S, \mathcal{V}),$$

whose composite we again write γ. We can form the trace

$$\operatorname{Tr}(\gamma|H^*(S, \mathcal{V})) =_{df} \sum (-1)^i \operatorname{Tr}(\gamma|H^i(S, \mathcal{V})).$$

Under suitable hypotheses, there will be a Lefschetz trace formula expressing this as a sum of local terms over the fixed points of the correspondence. A ***fixed point*** of the correspondence is a closed point t of T such that $\alpha(t) = \beta(t)$:

$$s \xleftarrow{\alpha} t \xrightarrow{\beta} s.$$

For such a point t, γ defines a map

$$\mathcal{V}_s = (\alpha^*\mathcal{V})_t \xrightarrow{\gamma_t} (\beta^*\mathcal{V})_t = \mathcal{V}_s,$$

on the stalks of $\mathcal{V}$.

Theorem C.1 (Lefschetz trace formula). *Assume that S is complete, that α is proper, that the set of fixed points of the correspondence is finite, and that each fixed point is of multiplicity one. Then*

$$\operatorname{Tr}(\gamma|H^*(S,\mathcal{V})) = \sum_t \operatorname{Tr}(\gamma_t|\mathcal{V}_{\beta(t)}),$$

where the sum is over the fixed points t of the correspondence.

Proof. See [Grothendieck et al 1977, III.4.12], and [Grothendieck 1977, 3.7].

Exercise C.2. To give a sheaf on a finite set endowed with the discrete topology is the same as to give a family of vector spaces indexed by the set. Prove the Lefschetz trace formula in the case that S and T are finite sets.

Zeta functions of complete varieties. Let S be a complete smooth variety over $\mathbb{F}_q$. The zeta function of S can be defined by either of the following two formulas:

$$Z(S,T) = \exp\left(\sum_{n>0} \nu_n(S)\cdot\frac{T^n}{n}\right) \tag{C.2.1}$$

where $\nu_n(S)$ is the number of points on S with coordinates in $\mathbb{F}_{q^n}$, or

$$Z(S,T) = \prod \det(1-FT|H^i(S\otimes\mathbb{F},\mathbb{Q}_\ell))^{(-1)^{i+1}}. \tag{C.2.2}$$

The equivalence of the two definitions follows from the Lefschetz trace formula applied to the correspondence F^n,

$$S \xleftarrow{F^n} S \xrightarrow{\text{id}} S, \quad \text{can}\colon F^{n*}\mathbb{Q}_\ell \to \mathbb{Q}_\ell,$$

which gives that

$$\operatorname{Tr}(F^n|H^*(S,\mathbb{Q}_\ell) = \sum_t \operatorname{Tr}(\text{id}\,|\mathbb{Q}_\ell) = \nu_n(S).$$

Both definitions are useful.

More generally, let $\mathcal{V}$ be a local system of $\mathbb{Q}_\ell$-vector spaces on S. Again there is a canonical morphism $F^*\mathcal{V}\to\mathcal{V}$, and Grothendieck's Lefschetz trace formula shows that the following two definitions of the zeta function of $\mathcal{V}$ on S are equivalent:

$$Z(S,\mathcal{V},T) = \exp\left(\sum_n \sum_{s\in S(\mathbb{F}_{q^n})} \operatorname{Tr}(F_s|\mathcal{V}_s)\cdot\frac{T^n}{n}\right),$$

or

$$Z(S,\mathcal{V},T)=\prod_i \det(1-FT|H^i(S\otimes\mathbb{F},\mathcal{V}))^{(-1)^{i+1}}.$$

(See [Milne 1980, VI.13].)

Now consider a complete smooth algebraic variety S over a number field E, and let $\mathcal{V}$ be a local system of $\mathbb{Q}_\ell$-vector spaces on S. The zeta function of $\mathcal{V}$ on S is defined to be the product of the local zeta functions,

$$Z(S,\mathcal{V},s)=\prod_v Z_v(S,\mathcal{V},s) \qquad \text{(product over all primes of } E\text{)}.$$

For a finite prime v where S and $\mathcal{V}$ have good reduction, i.e., where S reduces to a complete smooth algebraic variety $S(v)$ over $\kappa(v)$ and $\mathcal{V}$ reduces to an ℓ-adic local system $\mathcal{V}(v)$ on $S(v)$, the local zeta function is defined to be the zeta function of $\mathcal{V}(v)$ on $S(v)$:

$$Z_v(S,\mathcal{V},s)=Z(S(v),\mathcal{V}(v),q_v^{-s}), \quad q_v=[\kappa(v)].$$

Thus, for a good v,

$$\begin{aligned} Z_v(S,\mathcal{V},s)&=\exp\left(\sum_{n>0}\sum_{s\in S(\mathbb{F}_{q^n})}\operatorname{Tr}(F_s|\mathcal{V}_s(v))\cdot\frac{q_v^{-ns}}{n}\right)\\ &=\prod_i \det(1-Fq_v^{-s}|H^i(S(v)\otimes\mathbb{F},\mathcal{V}(v)))^{(-1)^{i+1}}. \end{aligned}$$

The proper smooth base change theorem in étale cohomology (ibid. VI.4.2) shows that for a good v there is a canonical isomorphism

$$H^i(S\otimes\mathbb{Q}^{\mathrm{al}},\mathcal{V})\approx H^i(S(v)\otimes\mathbb{F},\mathcal{V}(v)).$$

Consequently,

$$Z_v(S,\mathcal{V},s)=\prod_i \det(1-F_vq_v^{-s}|H^i(S\otimes\mathbb{Q}^{\mathrm{al}},\mathcal{V}))^{(-1)^{i+1}}$$

where now F_v denotes a geometric Frobenius element in $\operatorname{Gal}(\mathbb{Q}^{\mathrm{al}}/E)$.

In summary, $Z_v(S,\mathcal{V},s)$ can be defined in terms of action of $\operatorname{Gal}(\mathbb{Q}^{\mathrm{al}}/E)$ on the étale cohomology group $H^i(S\otimes\mathbb{Q}^{\mathrm{al}},\mathcal{V})$, and it can be computed in terms of the action of the Frobenius element on the fibres of $\mathcal{V}_v$ at the points on $S(v)$ with coordinates the fields $\mathbb{F}_{q^n}$.

Noncomplete varieties. Let S be a smooth variety over a finite field. When S is not complete, the two formulas (C.2.1) and (C.2.2) for the zeta function differ: Grothendieck's Lefschetz trace formula shows that the first definition gives

$$Z(S,T)_1 = \prod_i \det(1 - FT|H_c^i(S \otimes \mathbb{F}, \mathbb{Q}_\ell))^{(-1)^{i+1}},$$

where $H_c^i(S \otimes \mathbb{F}, \mathbb{Q}_\ell)$ denotes cohomology with compact support, and the second is

$$Z(S,T)_2 = \prod_i \det(1 - FT|H^i(S \otimes \mathbb{F}, \mathbb{Q}_\ell))^{(-1)^{i+1}},$$

Neither of these zeta functions has a functional equation. In fact, there is a duality between the cohomology with compact support and the ordinary cohomology, and it is the fact that these two cohomologies coincide when S is complete that gives the functional equation for Z(S,T).

Evidently, we need to find a definition that is intermediate between these two definitions. When S has a natural compactification $\bar{S}$ (not necessarily smooth) intersection cohomology with the middle perversity provides cohomology groups $IH^i(\bar{S}, \mathbb{Q}_\ell)$ that are self-dual and intermediate between $H^i(S, \mathbb{Q}_\ell)$ and $H_c^i(S, \mathbb{Q}_\ell)$; we define

$$Z(S,T) = \prod_i \det(1 - FT|IH^i(\bar{S} \otimes \mathbb{F}, \mathbb{Q}_\ell))^{(-1)^{i+1}}.$$

Note that this zeta function depends on the whole of $\bar{S}$, i.e., that the boundary of S in $\bar{S}$ contributes to $Z(S,T)$.

When S is a variety over a number field E with a natural compactification $\bar{S}$, then we can define

$$Z(S,s) = \prod Z_v(S,s)$$

where, for good primes,

$$Z_v(S,s) = \prod_i \det(1 - F_v q_v^{-s}|IH^i(\bar{S} \otimes \mathbb{Q}^{\mathrm{al}}, \mathbb{Q}_\ell)^{(-1)^{i+1}}.$$

The analogue of the proper smooth base change theorem for intersection cohomology shows that

$$Z_v(S,s) = Z(S(v), q_v^{-s}).$$

Let j be the open immersion $S \hookrightarrow \bar{S}$. The intersection cohomology of S is defined to be the hypercohomology of a certain complex IC. Let $j_!\mathbb{Q}_\ell$ be the extension by zero of the constant sheaf $\mathbb{Q}_\ell$ on S to $\bar{S}$. There are natural homomorphisms

$$j_!\mathbb{Q}_\ell \hookrightarrow Rj_*\mathbb{Q}_\ell \hookleftarrow IC$$

whose cokernels have homology supported on $\bar{S}-S$. Thus the trace of an operator on $IH^*(\bar{S},\mathbb{Q}_\ell) =_{df} \mathbb{H}^*(\bar{S}, IC)$ is the sum of the trace of the operator on $H_c^*(S,\mathbb{Q}_\ell) =_{df} H^*(\bar{S}, j_!\mathbb{Q}_\ell)$ with the trace of an operator on the cohomology group of a complex supported on the boundary. It is therefore natural to regard $Z(S,T)_1$ as being the contribution of S itself to the zeta function, and $Z(S,T)/Z(S,T)_1$ as being the contribution of the boundary.

In summary, $Z(S,T)_1$ can be regarded as the contribution of S itself to the zeta function, and it can be computed either in terms of the number of points of S with coordinates in $\mathbb{F}_q$ or in terms of the cohomology groups with compact support.

All of this applies to Shimura varieties. The natural compactification to take is the Baily-Borel compactification. The above discussion explains why, in computing the contribution of S itself to the zeta function, we need to compute the points on the reduction of S with coordinates in finite fields. In fact, we are interested in the zeta function of some summand of the intersection cohomology cut out by the Hecke operators, and to find the contribution of S to this zeta function one needs to compute the trace a Hecke operator times a power of the Frobenius endomorphism on the cohomology with compact support. For this we need Deligne's conjecture.

Deligne's conjecture. The Lefschetz trace formula is definitely false in general for a noncomplete variety (or a noncompact topological space). Consider for example the affine line (or the complex plane) and the map

$$\alpha\colon \mathbb{A}^1 \to \mathbb{A}^1, \quad x \mapsto x+1.$$

Clearly α has no fixed points, but it acts on $H^0(\mathbb{A}^1,\mathbb{Q}_\ell)$ as the identity map, and as the remaining cohomology groups are zero, the alternating sum of the traces is $1 \neq 0$. The result is the same if the cohomology groups with compact support are used: here $H_c^i(\mathbb{A}^1,\mathbb{Q}_\ell)=0$ for $i \neq 2$, and the trace of α on $H_c^2(\mathbb{A}^1,\mathbb{Q}_\ell)$ is 1. Note that the map extends to $\mathbb{P}^1$ and has a fixed point with multiplicity 2 at ∞, which is consistent with the fact that the traces of the map on $H^0(\mathbb{P}^1,\mathbb{Q}_\ell)$ and $H^2(\mathbb{P}^1,\mathbb{Q}_\ell)$ are both 1.

Nevertheless, Deligne conjectures the following. Let S be a smooth variety over an algebraically closed field k, and let $\mathcal{V}$ be an ℓ-adic local system on S ($\ell \neq \operatorname{char}(k)$). Consider a correspondence

$$S \xleftarrow{\alpha} T \xrightarrow{\beta} S, \quad \gamma\colon \alpha^*\mathcal{V} \to \beta^*\mathcal{V}.$$

When α is proper and β is finite, the correspondence defines a homomorphism $\gamma\colon H_c^i(S,\mathcal{V}) \to H_c^i(S,\mathcal{V})$ as before, and we write

$$\operatorname{Tr}(\gamma|H_c^*(S,\mathcal{V})) = \sum_i (-1)^i \operatorname{Tr}(\gamma|H_c^i(S,\mathcal{V})).$$

Also, for each fixed point t of the correspondence, we get a homomorphism

$$\gamma_t \colon \mathcal{V}_s \to \mathcal{V}_s, \quad s = \alpha(t) = \beta(t).$$

Now suppose that S and $\mathcal{V}$ are defined over a finite field. Then, as we noted above, there is a canonical isomorphism $F^*\mathcal{V} \to \mathcal{V}$, and we can compose the original correspondence with F^r to get a new correspondence

$$S \xleftarrow{\alpha} T \xrightarrow{F^r \circ \beta} S, \quad \alpha^*\mathcal{V} \xrightarrow{\gamma} \beta^*\mathcal{V} \to \beta^* F^{r*}\mathcal{V},$$

which we denote $\gamma^{(r)}$.

Conjecture C.3 (Deligne). *There exists an r_0 such that for all $r \geq r_0$,*

$$\mathrm{Tr}(\gamma^{(r)} | H_c^*(S, \mathcal{V})) = \sum_t \mathrm{Tr}(\gamma_t^{(r)} | \mathcal{V}_{\alpha(t)}),$$

where the sum is over the fixed points t of the correspondence $\gamma^{(r)}$, i.e., the set of points t such that $\alpha(t) = F^r\beta(t)$.

In fact, Deligne's conjecture is more general than we have stated it—he does not require S or $\mathcal{V}$ to be smooth, and he allows β to be quasi-finite.

Theorem C.4. *Assume that resolution of singularities holds; then Conjecture C.3 is true.*

Proof. This has been proved, independently and almost simultaneously, by Pink and Shpiz—see [Pink 1990] and [Shpiz 1990]. □

Since resolution of singularities is not known in characteristic $p \neq 0$, this result is not useful as stated. However, the results of Pink and Shpiz are much more explicit—for example, in the case that $\mathcal{V}$ is the constant sheaf, they state that Conjecture C.3 is true provided S can be realized as the complement of a normal crossings divisor in a smooth compactification.

Exercise C.5. Verify the conjecture for the correspondence

$$\mathbb{A}^1 \xleftarrow{\alpha} \mathbb{A}^1 \xrightarrow{\mathrm{id}} \mathbb{A}^1, \quad \alpha^*\mathbb{Q}_\ell = \mathbb{Q}_\ell \xrightarrow{\mathrm{id}} \mathbb{Q}_\ell,$$

where $\alpha = x \mapsto x + 1$.

Shimura varieties. Let $\mathrm{Sh}_K(G,X)$ be a Shimura variety, initially considered over $\mathbb{C}$, and let $\xi\colon G \to \mathrm{GL}(V)$ be a representation of G on a finite dimensional $\mathbb{Q}$-vector space. This gives rise in a natural way to a local system of $\mathbb{Q}$-vector spaces $\mathcal{V} = \mathcal{V}_K$ on $\mathrm{Sh}_K(G,X)$ (for the complex topology). For any $g \in G(\mathbb{A}_f)$ we get a correspondence on $(\mathrm{Sh}_K(G,X), \mathcal{V}_K)$:

$$\mathrm{Sh}_K(G,X) \xleftarrow{g} \mathrm{Sh}_{K'}(G,X) \to \mathrm{Sh}_K(G,X)$$

where $K' = gKg^{-1} \cap K$ and the second map is the obvious quotient map. When we "tensor" $\mathcal{V}$ with $\mathbb{Q}_\ell$, we obtain a local system of $\mathbb{Q}_\ell$-vector spaces $\mathcal{V}_K = \mathcal{V}_K(\mathbb{Q}_\ell)$ for the étale topology on $\mathrm{Sh}_K(G,X)$, and the sheaf and the correspondence is defined over $E(G,X)$. Write $\mathcal{T}(g)$ for this correspondence.

Write $\mathrm{Sh}_K(G,X)(v)$ for the reduction of $\mathrm{Sh}_K(G,X)$ modulo a prime v of $E(G,X)$, and set $S_K(v) = \mathrm{Sh}_K(G,X)(v) \otimes \mathbb{F}$. Let $\mathcal{V}(v)$ be the sheaf on $S_K(v)$ defined by $\mathcal{V}$. For a sufficiently good prime, $\mathcal{T}(g)$ will define a correspondence on the reduction, and we wish to compute

$$\mathrm{Tr}(\mathcal{T}(g)^{(r)}|H_c^*(\mathrm{S}_K(v), \mathcal{V}(v))).$$

Theorem C.6. *Assume that $Sh_K(G,X)$, $\mathcal{V}$, and some smooth toroidal compactification of $Sh_K(G,X)$ have good reduction at a prime v. There exists an r_0 such that for $r \geq r_0$, and any $g \in G(\mathbb{A}_f^p)$,*

$$\mathrm{Tr}(\mathcal{T}(g)^{(r)}|H_c^*(S_K(v), \mathcal{V}(v))) = \sum_{t'} \mathrm{Tr}(\mathcal{T}(g)^{(r)}|\mathcal{V}_t) \qquad \text{(C.6.1)}$$

where t' runs over the set of points in $Sh_{K'}(G,X)(\mathbb{F})$ such that $g(t') = F^r(t')$ (in $Sh_K(G,X)(\mathbb{F})$) and $t = g(t')$.

Proof. This follows from the more precise form of the theorem of Pink and Shpiz. □

The object of the main body of the article is to compute the term on the right of (C.6.1); by an abuse of notation, we denote it by

$$\textstyle\sum_{t'} \mathrm{Tr}(\mathcal{T}(g)^{(r)}|\mathcal{V}_t(\xi)).$$

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